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# Qualitative and geometric methods for large econometric models *

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This study is a condensed and revised version of two previous papers, one by Ritschard [17] on computable qualitative comparative statics, the other by Rossier [21] on a geometric approach of economic modeling. Since qualitative methods deal with open intervals represented by signs (+,−), and the proposed geometric methods are concerned with closed intervals, the main purpose of this unified version is to give an overview of the knowledge one can derive when the basic information about the parameters of a large economic model is given by intervals. In this general framework, quantitative information derived from econometric models appears as a particular case of closed intervals.

However, it remains a conceptual distinction between qualitative and geometric approaches, coming from the treatment of exogenous variables, which are considered like parameters in geometric formulation. For this reason, Section 1 of this paper emphasizes qualitative methods in comparative static analysis, Section 2 being devoted more specifically to geometric methods applicable to linearized models.

## 0. Introduction

Broadly speaking, comparative static analysis is concerned with the study of how, in a given economic model, endogenous variables react to designated changes in the level of exogenous variables or parameters. In the case of a fully quantified model, such an analysis may be carried out by solving it numerically once before and once after the considered change. Proceeding so, as well as looking into a quantified impact multipliers matrix, will not allow, however, to distinguish between the conclusions stemming out from the particular empirical content of the model and those emerging from its theoretical background. This is precisely the aim of qualitative and geometric methods which, by dealing with the model in its general formulation, take into account only robust information, i.e. the invariant one, in order to determine its logical implications.

For marginal changes, one way to achieve a qualitative comparative static analysis is to develop formally the impact multipliers in terms of the derivatives of the model relations. This should allow us to see, for instance, to what extent a priori information about the derivatives would suffice to determine the range of some impact multipliers. Such an analytical approach is generally used to determine qualitative properties of simple macro-models. [1] It also leads to fruitfull results when applied to models, such as expenditures systems, issued from som underlying optimization problem. Let us notice however that the richness of the results in this last case follows from the fact that the model's relation derivatives can themselves be expressed in terms of the second order derivatives of the objective function. [2] Nevertheless, as the size of the model increases, i.e. the number of relations, it becomes rapidly tedious or even impossible to write down such formal expansions of the impact multipliers. A computer, for instance, would be of few help.

An alternative approach to comparative static problems consists in studying systematically the implication of some specific kind or family of given informations. This is the approach retained in this paper which is devoted to the study of what can be said from the knowledge of the intervals of the model relations derivatives. More specifically,

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[1] Examples can be found, for instance in [7, pp. 168–177] or in [24].

[2] These second order derivatives have naturally to satisfy the second order conditions of the considered optimizing problem.

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efficient tools for analyzing this kind of information will be proposed.

Among the specific literature on qualitative methods, most of the studies aim at determining the general conditions under which significant conclusions will follow from the qualitative assumptions. Hence they don't set up efficient tools for analyzing given models. Along this line, one can quote the standard forms of the Jacobian brought into focus by Lancaster [10,11] and Gorman [8]. Other important studies use cycles and chains algebra (see for instance [2]). An excellent survey of all these studies is found in [1], (see also [19]). The pioneering work of Samuelson [22] who established the foundations of qualitative calculus, and the developments of this calculus by Lancaster [12,13] are, as far as we know, the only studies in which tools for a systematical analysis of qualitative assumptions are proposed. [3] The suggested techniques are based, however, on an elimination principle for which the starting set increases exponentially with the size of the analyzed model (see Section 1.3 hereafter). The applicability of these tools remains therefore very limited, and this makes, from our point of view, the relevance of developing more efficient technics applicable to large economic models, like those which are outlined in Section 1.

In Section 1.1, we will formalize the problem and show why comparative static analysis should always begin with a study of the causal ordering of the model. In Section 1.2, an efficient algorithm for determining the so called *qualitatively linked* variables will be proposed. In Section 1.3, we will show how the Samuelson–Lancaster qualitative calculus can be improved by means of a branch-and-bound procedure. Section 1.4 illustrates an application of these technics to an economic model, which, for pedagogic purposes, has been choosen quite small.

The geometric approach of qualitative conclusions with open intervals or signs has been made by Lancaster [12] from a theoretical point of view, and practically it leads to the qualitative calculus reported in Section 1. Hence Section 2 is only concerned with closed intervals. It aims to point out some main properties of the geometric figures generated, in the space of endogenous variables, by sets of intervals interconnected in linear relations. Such figures will be called polytopes by analogy to the well-known convex polytope notion (see for instance [9]), commonly used in linear programming and more recently in data analysis (see for instance [15]). But to our best knowledge of the economic literature, the definition and study of the polytopes reported in Section 2 were first introduced in [20, Chapter IX] and developed for algorithmic issues in [21].

Section 2.1 defines the methodology adopted while some main properties of polytopes are given in Section 2.2. Section 2.3 is devoted to the projection of polytopes on selected axes and planes. The two last sections deal with algorithmic remarks, an illustration and some practical considerations on the interpretation of the projections.

Finally, our view of the future of qualitative and geometric methods will help to conclude.

[3] Analysis of macro-models by means of these tools can be found in [19] and [23].

## 1. The qualitative approach

### 1.1. Causal analysis and qualitative calculus

Let us consider an economic model formally represented by a system of $m$ relations:

$$h(y,z)=0 \tag{1.1}$$

which relates the equilibrium level of the $m$ endogenous variable $y$ to the state of the environment represented by the $k$ exogenous variables $z$.

By giving the sign (positive, negative or zero) of each element of the matrix

$$\left[\frac{\partial h}{\partial y'} \;\middle|\; \frac{\partial h}{\partial z'}\right], \tag{1.2}$$

we define the so called *qualitative structure* of the model. The impact multipliers matrix, in which one is interested when doing comparative statics, can be expressed in terms of the two matrices in (1.2):

$$\frac{\partial y}{\partial z'}=-\left(\frac{\partial h}{\partial y'}\right)^{-1}\frac{\partial h}{\partial z'}. \tag{1.3}$$

Thus the problem considered is that of providing information about the content of the matrix (1.3) when only the knowledge of the qualitative structure is taken into account.

Obviously this information can only concern the sign content of the matrix $\partial y/\partial z'$. The most significant result one can obtain for a multiplier $\partial y_i/\partial z_j$ will be, therefore, the unambiguous determination of its sign. If so, the multiplier $\partial y_i/\partial z_j$ is called *qualitatively determined*. It means that its sign (positive, negative, or zero) is determined regardless of the numerical value that might take the non-zero elements of the matrix (1.2) (provided that these values are compatible with the qualitative structure analyzed).

If a multiplier $\partial y_i/\partial z_j$ is not qualitatively determined, then its sign may be positive, negative, or zero depending on the particular numerical determination of the matrix (1.2). Note that in this case the qualitative structure allows for a *possible zero* value of $\partial y_i/\partial z_j$. It is important to distinguish this kind of possible zeros from *qualitative zeros*, where the latter correspond to qualitatively null determined multipliers, i.e. multipliers which are identically null for the given qualitative structure.

We first deal with the setting up of these qualitative zeros. A multiplier is identically null:

$$\partial y_i/\partial z_j \equiv 0, \tag{1.4}$$

if and only if in the model (1.1) $y_i$ is determined independently from the exogenous variables $z_j$, i.e. if and only if $z_j$ has no causal effect on $y_i$. Thus the problem is here simply to determine, for each exogenous variable $z_j$, the set of endogenous variables which it does not influence. This can easily be done through a causal analysis of the model, for which efficient tools related to graph theory have recently been developed. (See for instance [4]; for a computer program see [5,6,18].)

Let us just recall that the *causal outline* of a model of type (1.1) corresponds to its block recursive decomposition. It is characterized by the block triangular form into which the matrix $\partial h/\partial y'$ can be transformed through independent permutations of rows and columns. Let

$$D=\begin{bmatrix} D_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ D_{p1} & \cdots & D_{pp} \end{bmatrix}, \tag{1.5}$$

where the diagonal submatrices $D_{ii}$ are square and irreducible, be this block triangular matrix. The relations $h^k(y,z)=0$ corresponding to the rows of a block $D_{kk}$ are the smallest subset of relations which determines the endogenous variables $y^k$ corresponding to the columns of $D_{kk}$. Indeed all variables $y^j, j<k$, can be considered as exogenous in $h^k(y,z)=0$ whereas the variables $y^j, j>k$, don't appear in the relations $h^k(y,z)=0$.

By considering this decomposition, it can be shown that a variable $y_i$ of $y^k$ is not influenced by an exogenous variable $z_s$ if and only if

$$\begin{aligned}&\text{(i) } \partial h^k/\partial z_s = 0 \text{ and}\\ &\text{(ii) } \partial h^j/\partial z_s = 0 \text{ for all } j \text{ for which}\end{aligned} \tag{1.6}$$

one can find a sequence of non-zero matrices

$$D_{kr_1}, D_{r_1r_2}, \ldots, D_{r_rj}.$$

Note that these conditions, which can easily be used to determine the qualitative zeros of $\partial y/\partial z'$, are based only upon the knowledge of the zero entries of (1.2). Qualitative zeros are therefore more general properties of a model than those studied in the remainder of the paper.

For a generic exogenous variable $\alpha = z_s$, the corresponding multipliers vector $\partial y/\partial\alpha$ is a solution of the following linear system:

$$\frac{\partial h}{\partial y'}\frac{\partial y}{\partial \alpha} = -\frac{\partial h}{\partial \alpha}. \tag{1.7}$$

From the condition (1.6), if $\partial y^1/\partial\alpha$ is the subvector of all qualitative zeros of $\partial y/\partial\alpha$, then (1.7) can be written as follows through suitable permutations:

$$\begin{bmatrix} \dfrac{\partial h^1}{\partial y^{1'}} & 0 \\ \dfrac{\partial h^2}{\partial y^{1'}} & \dfrac{\partial h^2}{\partial y^{2'}} \end{bmatrix}\begin{bmatrix} \dfrac{\partial y^1}{\partial \alpha} \\ \dfrac{\partial y^2}{\partial \alpha} \end{bmatrix} = \begin{bmatrix} 0 \\ -\dfrac{\partial h^2}{\partial \alpha} \end{bmatrix} \tag{1.8}$$

where none of $\partial y^2/\partial\alpha$ components is a qualitative zero. Since

$$\frac{\partial h^2}{\partial y^{1'}}\frac{\partial y^1}{\partial \alpha} \equiv 0, \tag{1.9}$$

$\partial y^2/\partial\alpha$ is a solution of the subsystem

$$\frac{\partial h^2}{\partial y^{2'}}\frac{\partial y^2}{\partial \alpha} = -\frac{\partial h^2}{\partial \alpha}, \tag{1.10}$$

which doesn't give rise to qualitative zeros. From here on we shall thus, without loss of generality, only consider systems without qualitative zeros.

By considering the following qualitative matrix and vector:

$$\begin{aligned}&A=(a_{ij}), \quad i,j=1,\ldots,m,\\ &b=(b_i), \quad i=1,\ldots,m\end{aligned} \tag{1.11}$$

with

$$a_{ij} = + \Leftrightarrow \frac{\partial h_i}{\partial y_j} \geqslant 0, \qquad b_i = + \Leftrightarrow \frac{\partial h_i}{\partial \alpha} \geqslant 0,$$

$$a_{ij} = - \Leftrightarrow \frac{\partial h_i}{\partial h_j} \leqslant 0, \qquad b_i = - \Leftrightarrow \frac{\partial h_i}{\partial \alpha} \leqslant 0,$$

$$a_{ij} = 0 \Leftrightarrow \frac{\partial h_i}{\partial y_j} \equiv 0, \qquad b_i = 0 \Leftrightarrow \frac{\partial h_i}{\partial \alpha} \equiv 0,$$

and the following obvious qualitative operations:

| *sum* | + | − | 0 | r |
|---|---|---|---|---|
| + | + | r | + | r |
| − | r | − | − | r |
| 0 | + | − | 0 | r |
| r | r | r | r | r |

| *product* | + | − | 0 | r |
|---|---|---|---|---|
| + | + | − | 0 | r |
| − | − | + | 0 | r |
| 0 | 0 | 0 | 0 | 0 |
| r | r | r | 0 | r |

where an r stands for undetermination, we have thus to face the problem of solving a qualitative system of the form

$$Ax = -b. \tag{1.12}$$

By setting

$$H = (b \mid A), \qquad v' = (+ \mid x'), \tag{1.13}$$

we can equivalently consider the more convenient homogenous form of (1.12):

$$Hv = 0. \tag{1.14}$$

A qualitative vector $s$ is a solution of (1.14) if and only if it satisfies

$$h'_{i.}s = \Sigma h_{ij}s_j = 0 \text{ or r}, \quad i = 1,2,\ldots,m, \tag{1.15}$$

where $h'_{i.}$ stands for the $i$th row of $H$. Such a solution corresponds to a sign content of $\partial y/\partial\alpha$ compatible with the qualitative structure considered.

In general a qualitative system will admit more than one solution. Let us denote by $S$ the set of all possible solutions with $s_1 = +$ according to (1.13). A multiplier $\partial y_i/\partial\alpha$ will be qualitatively determined if the corresponding element $s_{i+1}$ has the same sign in each $s$ of $S$. For instance all components of $\partial y/\partial\alpha$ will be qualitatively determined if and only if card$(S) = 1$, i.e., if and only if (1.14) admits a unique solution.

More generally we will say that two variables $v_i$ and $v_j$ are *qualitatively linked* if and only if $s_i$ and $s_j$ are always of the same sign, or always of opposite signs in all $s$ of $S$. In the first case they will be called *positively linked* and in the second *negatively linked*. Remembering (1.13) qualitatively determined variables will be for instance qualitatively linked with $v_1$.

Qualitatively linked variables can easily be determined by examining the set $S$. The difficulty is, however, precisely to obtain this set $S$. Before studying this problem in Section 1.3 we show in Section 1.2 how qualitatively linked variables can be set up directly. As a byproduct a qualitative aggregation procedure is proposed, which allows to reduce the size of the qualitative system being analysed.

### *1.2. Qualitatively linked variables and qualitative aggregation*

It can be shown that the relation $v_i L v_j$ "$v_i$ is qualitatively linked with $v_j$" defined on the $p$ variables $v$ of an $r$ relations qualitative system[4]

$$Hv = 0, \tag{1.16}$$

is an equivalence (reflexive, symmetric and transitive) relation. According to it the $p$ variables $v$ can be partitioned into equivalence classes.

The qualitative link, positive or negative, between each pair of elements of a group of qualitatively linked variables $v^1$ can be summarized simply by a sign vector $q^1$, of the same size as $v^1$, by setting

$$q_i = q_j \Leftrightarrow v_i L_+ v_j$$
$$q_i = -q_j \Leftrightarrow v_i L_- v_j \tag{1.17}$$

where $L_+$ stands for "is positively linked with" and $L_-$ for "is negatively linked with". With respect to the set $S$ of solutions without zero of (1.16), $q^1$, or $-q^1$, is the subvector $s^1$ corresponding to $v^1$ in any solution $s$ of $S$.

The knowledge of the qualitative link $q^1$ for a class $v^1$ together with that of the sign $s_i$ of only one component $v_i$ of $v^1$ is sufficient to determine the signs $s^1$ of all variables $v^1$. Assuming $q_i = +$, we have

$$s^1 = s_i q^1. \tag{1.18}$$

[4] Note that no special assumption is made about the relative size of $p$ and $r$.

One can think therefore to reduce the size of the system (1.16) by considering only one variable of $v^1$. Since all variables do not appear in all relations, one way to preserve the full qualitative information available about $v^1$ is to replace the subvector $v^1$ of order $p_1$ by the *qualitative aggregate*:

$$v_1^* = q^{1'}v^1. \qquad (1.19)$$

Assuming without loss of generality that $v' = (v^{1'} \; v^{2'})$, the full vector $v$ is thus replaced by

$$v^* = \begin{bmatrix} q^{1'}v^1 \\ v^2 \end{bmatrix} = Mv \qquad (1.20)$$

where the $(p - p_1 + 1) \times p$ qualitative *aggregation matrix* $M$ is defined as

$$M = \left[\begin{array}{c|cccc} q^{1'} & & 0' & & \\ \hline & + & & & 0 \\ 0 & & + & & \\ & & & \ddots & \\ & 0 & & & + \end{array}\right]. \qquad (1.21)$$

Since the ordering of the matrix $H$ columns corresponds to that of the variables $v$, one can consider aggregating $H$ into the following $r \times (p - p_1 + 1)$ matrix:

$$HM' = (H_{.1} \mid H_{.2})\, M' = (H_{.1}q^1 \mid H_{.2}). \qquad (1.22)$$

Fore some rows $h'_{i.} = (h^{1'}_{i.} \mid h^{2'}_{i.})$ of $H$, the product $h^{1'}_{i.}q^1$ might be, however, qualitatively undetermined. According to (1.15) the corresponding relations $h'_i v = 0$ admit as solution any sign vector $s$ compatible with the qualitative link $q^1$, i.e., all $s$ for which $s^1 = \pm q^1$. Therefore, these rows $h'_{i.}$ don't provide any supplementary information and can be eliminated.

If we designate by $H^{**}$ the matrix resulting from the suppression of the rows $h'_{i.}$ for which $h^{1'}_{i.}q^1$ is undetermined, the *aggregated matrix* $H^*$ will be

$$H^* = H^{**}M'. \qquad (1.23)$$

It can be shown (see [16, pp. 103–104]) that there is a one to one correspondance between $S$ and $S^*$, where $S$ is the set of solutions without zero of (1.16) and $S^*$ the set of solutions without zero of the aggregated system

$$H^*v^* = 0 \qquad (1.24)$$

where $v^*$ and $H^*$ are defined in (1.20) and (1.23). Indeed we have

$$S = \{s \mid s = M's^*, s^* \in S^*\},$$
$$S^* = \{s^* \mid s^* = Ms, s \in S\}. \qquad (1.25)$$

All what has been said so far can easily be generalized for the case of several groups of qualitatively linked variables. Let $q^i$, $i = 1, \ldots, K$ be the qualitative links for $K$ groups $v^i$. To generalize the aggregation rules (1.20) and (1.23), we simply have to define the new aggregation matrix

$$M = \left[\begin{array}{cccc|cccc} q^{1'} & & 0 & & & & & \\ & q^{2'} & & & & & & \\ & & \ddots & & & 0 & & \\ 0 & & & q^{K'} & & & & \\ \hline & & & & + & & & 0 \\ & & & & & + & & \\ & 0 & & & & & \ddots & \\ & & & & 0 & & & + \end{array}\right]. \qquad (1.26)$$

The matrix $H^{**}$ in (1.23) will be obtained by eliminating from $H$ the rows $h'_{i.}$ for which one of the products $h^{j'}_{i.}q^j$, $j = 1,2,\ldots,K$ is qualitatively undetermined.

It is important to notice here that this elimination of rows from H can lead to a matrix $H^*$ with colums of zeros. In case all columns of $H^{**}$ corresponding to a class $v^i$ of qualitatively linked variables are null, the column of $H^*$ corresponding to the aggregate $v_i^* = v^{i'}q^i$ will also be null from (1.23). If so, the aggregate $v_i^*$ might be $+$ or $-$ regardless of the sign of the other components of $v^*$, and will be called an *independent variable* or *aggregate*, and the class $v^i$ represented by $v_i^*$ an *independent class.*

Assuming $v^{*\prime} = (v^{\Delta'} \mid \bar{v}^{\Delta'})$ where $\bar{v}^{\Delta}$ is a subvector of $k$ independent variables, the set $S^*$ of solutions of the aggregated system $H^*v^* = 0$ can easily be obtained from the set $S^{\Delta}$ of solutions of

$$H^{\Delta}v^{\Delta} = 0, \qquad (1.27)$$

where $H^{\Delta}$ results from the suppression of the columns corresponding to $\bar{v}^{\Delta}$ in $H^*$. Since all elements of $\bar{v}^{\Delta}$ are independent variables the $2^k$ possible sign combinations of length $k$ are admissible for $\bar{v}^{\Delta}$, and thus each solution $s^{\Delta}$ will give rise to $2^k$ solutions of the form

$$s^* = (s^{\Delta} \mid \bar{s}^{\Delta}).$$

We now turn to the setting up of the qualitatively linked variables which indeed must be known before one can proceed to a qualitative aggregation.

The method we propose is an iterative one which makes use of the qualitative aggregation principle. First a pair of qualitatively linked variables is sought and, if found, the system is aggregated according to it. The next step consists in looking for a pair of qualitatively linked variables for the aggregated system and so on, until a system without qualitatively linked variables is obtained.

In order to set up at each step a couple of linked variables, one can make use of the following very simple, but only sufficient, condition.

**Sufficient condition 1** (S.C.1). *If $h_{ki}$ and $h_{kj}$ are the only two non-zero elements of a row $h'_{k\cdot}$ of $H$, then the variables $v_i$ and $v_j$ of $Hv=0$ are qualitatively linked.*

The proof of this result is straightforward since, if $h_{ki}$ and $h_{kj}$ are the only non-zero elements of $h'_{k\cdot}$, we have

$$h'_{k\cdot}v=0 \Leftrightarrow v_i=\frac{-h_{kj}}{h_{ki}}v_j. \tag{1.28}$$

It follows also from (1.28) that the qualitative link between $v_i$ and $v_j$ is given by $q'=(h_{ki} \quad -h_{kj})$, i.e.,

$$\begin{aligned} &h_{ki}=h_{kj}\neq 0, \quad h_{ks}=0, \quad s\neq i,j \quad \Rightarrow v_i L_- v_j, \\ &h_{ki}=-h_{kj}\neq 0, \quad h_{ks}=0, \quad s\neq i,j \quad \Rightarrow v_i L_+ v_j. \end{aligned} \tag{1.29}$$

The following example shows that condition S.C.1 is not necessary:

$$Hv=\begin{bmatrix} + & + & + \\ + & + & - \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{1.30}$$

This qualitative system admits only two normalized solutions ($v_1=+$):

$s(1)'=(+ \quad - \quad +)$ and $s(2)'=(+ \quad - \quad -)$

from which obviously $v_1$ and $v_2$ are negatively linked, even if each row of $H$ has more than two non-zero entries. In fact, the system (1.30) is equivalent, with respect to the solutions without zero, to the single qualitative relation

$$(+ \quad + \quad 0)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}=0.$$

When condition S.C.1 is not fulfilled, one can think of checking if $H$ can be transformed into an equivalent qualitative matrix satisfying S.C.1. This can be done for instance by means of the following sufficient condition (see [16, Chapter 4, Section 4.1]).

**Sufficient condition 2** (S.C.2). *If two rows $h'_{i\cdot}$ and $h'_j$ of $H$ can be written in the following form by joint permutations of their elements:*

$$h'_{i\cdot}=\left(h^{1\prime}_{i\cdot} \quad h_{ik} \quad h^{2\prime}_{i\cdot}\right)$$

$$h'_{j\cdot}=\left(h^{1\prime}_{j\cdot} \quad h_{jk} \quad 0'\right)$$

*with*

$$\begin{aligned} &h^1_{j\cdot}=h^1_{i\cdot} \quad \text{and} \quad h_{jk}=0 \text{ or } -h_{ik}, \quad \text{or} \\ &h^1_{j\cdot}=-h^1_{i\cdot} \quad \text{and} \quad h_{jk}=0 \text{ or } h_{ik} \end{aligned} \tag{1.31}$$

*then $h_{ik}$ can be replaced by a zero without affecting the set $S$ of solutions without zero of $Hv=0$.*

The following special cases can be set up.

(i) If $h_{jk}=0$ in (1.31), all elements of $h^2_{i\cdot}$ can be replaced by zeros. Since the transformed row will thus provide the same qualitative information as $h'_{j\cdot}$, it can simply be deleted from $H$,

(ii) If $h^2_{i\cdot}=0$, we have a situation analogous to that of (1.30), i.e., the two rows differ in only one non-zero element or in all but one. In this case $h_{ik}$ and $h_{jk}$ can both be replaced by zero and the two transformed rows will provide the same qualitative information. The couple of rows in (1.31) will thus be equivalent to the single row $(h^{1\prime}_{i\cdot} \mid 0')$.

Condition S.C.2 is an operational one since it can be checked simply by comparing two by two the rows of $H$. Consider for example the following matrix:

$$H=\begin{bmatrix} + & - & 0 & - & 0 \\ + & - & - & + & 0 \\ + & - & + & 0 & - \\ - & + & 0 & 0 & - \end{bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \tag{1.32}$$
$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix}$$

From rows 1 and 2 we can set $h_{24}=0$, and from rows 3 and 4 $h_{35}=0$. Thus $H$ is equivalent to the qualitative matrix

$$\begin{bmatrix} + & - & 0 & - & 0 \\ + & - & - & 0 & 0 \\ + & - & + & 0 & 0 \\ - & + & 0 & 0 & - \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \qquad (1.33)$$

in which the rows 2 and 3 can be replaced by

$$(+ \quad - \quad 0 \quad 0 \quad 0). \qquad (1.34)$$

Finally it can be shown that $h_{14}$ and $h_{45}$ might also be zero, and thus $H$ is equivalent to the row (1.34). Since (1.34) fulfills condition S.C.1, it comes that $v_1$ and $v_2$ are positively linked in $Hv = 0$. By aggregating (1.34) according to this link no more qualitative system will subsist. Thus, we conclude that the variables $v$ can be partitioned into independent classes, i.e.: $\{v_1, v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}$.

Both conditions S.C.1 and S.C.2 are only sufficient. Nevertheless, together with the qualitative aggregation principle, they should allow us to set up almost all qualitative links. Indeed qualitative links which could not be detected by means of S.C.1 and S.C.2 are singular cases. On the other hand it would be excessively time consuming to check for the absence of such singular cases. Thus, in order to preserve efficiency, it is preferable to neglect these singular cases and to limit oneself to the algorithm reflected by Fig. 1.

Concerning this diagram it must be noticed that we call *equivalent quasi-minimal* a matrix obtained from $H$ by annuling successively all elements which might be zero according to S.C.2.

It must also be noticed that the qualitative link $q^1$ for each class of qualitatively linked variables $v^1$ is built step by step. Without loss of generality suppose that a qualitative link between two aggregates $v_i^*$ and $v_j^*$ has been set up at the $k$th step. Assuming $q^i$ and $q^j$ are respectively the links between the variables $v^i$ represented by $v_i^*$ and $v^j$ represented by $v_j^*$, the link $q^1$ for the vector $v^{1\prime} = (v^{i\prime} \mid v^{j\prime})$ will be

$$q^{1\prime} = \begin{cases} (q^{i\prime} \mid q^{j\prime}) & \text{if} \quad v_i^* L_+ v_j^*, \\ (q^{i\prime} \mid -q^{j\prime}) & \text{if} \quad v_i^* L_- v_j^*. \end{cases} \qquad (1.35)$$

As described in Fig. 1 the algorithm has been computerized and successfully tested with up to 200 relations systems.[5]

[5] The algorithm has been implemented in a new version of the program ANAS [18].

### 1.3. Improvement of Samuelson–Lancaster's elimination principle

This section is devoted to the computation of the qualitative solutions set $S$ for an $r$ relations and $p$ variables qualitative system:

$$\underset{(r \times p)}{H} v = 0. \qquad (1.36)$$

We shall assume only that no variable $v_i$ is a qualitative zero so that $S$ contains at least one solution $s$ without zero.

To solve this problem, one may use Samuelson's elimination principle [22, pp. 23–29]. This technique consists in eliminating from the set of all sign combinations $s$ of length $p$ those for which (1.15) does not hold. It leads thus to compare $2^p$ a priori possible sign vectors $s$ with each row of $H$.[6] In order to improve this procedure Lancaster (1966) suggested to represent sign vectors by binary numbers according to which the comparison of sign vectors reduces simply to the comparison of integer numbers.

Under this form the efficiency of the elimination principle remains limited, however, by the size $2^p$ of the *reference set*, i.e. the set of all a priori sign combinations which increases exponentially with $p$, the number of variables. For example, one would have to test more than $10^6$ possibilities for $p = 20$ and more than $10^{15}$ for $p = 50$.

With respect to this limitation a gain in efficiency will be achieved if before computing $S$ we can reduce the number $p$ of variables by a qualitative aggregation. Indeed from (1.25) this can be done without loss of generality. Nevertheless, the method will fail when the number of qualitatively linked variables classes remains too large.

By means of a decomposition of the qualitative system (1.36) the elimination principle may become operational even for large $p$. Assuming $H$ contains zeros we can, through suitable permutations, write (1.36) into the following form:

$$\begin{bmatrix} H_{11} & & & 0 \\ H_{21} & H_{22} & & \\ & & \ddots & \\ H_{l1} & & & H_{ll} \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ \vdots \\ v^l \end{bmatrix} = 0 \qquad (1.37)$$

[6] Note that there are $2^p$ a priori possibilities if only sign vectors without zero are considered. Otherwise this number would be $3^p$.

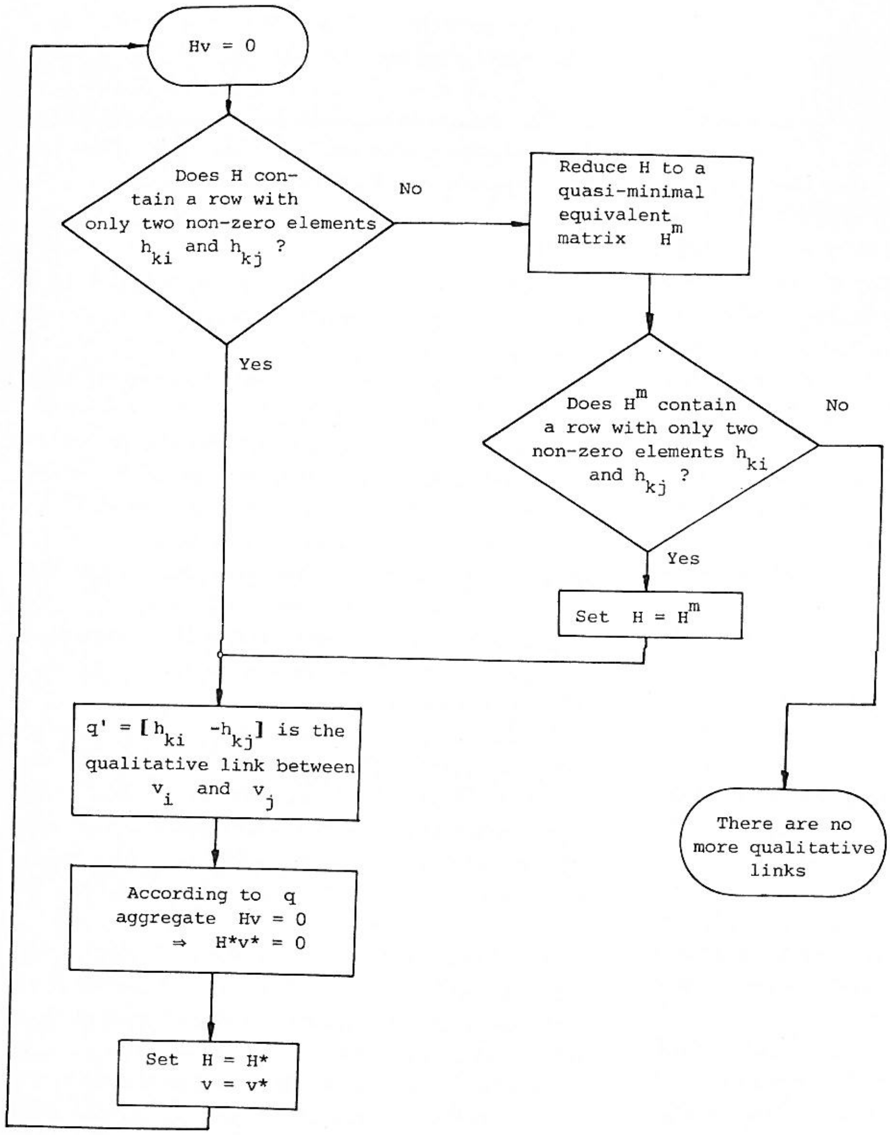



Fig. 1. The iterative qualitative aggregation procedure.

where $l$ is assumed to be as large as possible, which implies for instance that the last column of each $r_i \times p_i$ submatrix $H_{ii}$ contains only non-zero elements. Obviously the possible sign combinations $s^1$ for $v^1$ must belong to the qualitative solutions set $S^1$ of

$$H_{11}v^1 = 0. \tag{1.38}$$

Likewise the possible sign combinations $(s^{1'} \mid s^{2'})$ for $(v^{1'} \mid v^{2'})$ must be qualitative solutions of

$$(H_{21} \mid H_{22})\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} = 0. \tag{1.39}$$

Since $v^1$ in (1.39) must also be compatible with (1.38), one needs only to seek the solutions of (1.39) for which $s^1$ is a qualitative solution of (1.38). Thus there are only $t2^{p_2}$ a priori possibilities to be considered when (1.38) admits $t$ solutions. Seeking recursively, along the same way the solutions of

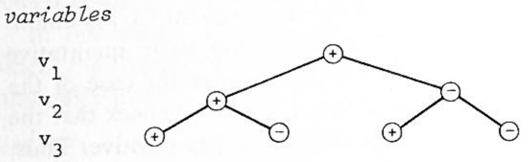
$$(H_{kl} \cdots H_{kk})\begin{bmatrix} v^1 \\ \vdots \\ v^k \end{bmatrix} = 0, \tag{1.40}$$

among the sign combinations $(s^{1'} \cdots s^{k-1'}\ s^{k'})$ where $(s^{1'} \cdots s^{k-1'})$ is one of the solutions set up at the previous step, one will finally at the $l$th step, obtain the set $S$ of qualitative solutions for the whole system (1.37).

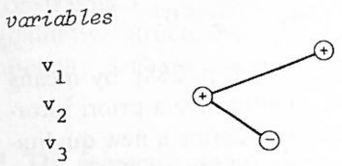
To illustrate this recursive procedure let us consider the following example:

$$\left[\begin{array}{ccc|c|c} - & + & - & 0 & 0\\ 0 & + & + & 0 & 0\\ \hline - & 0 & + & + & 0\\ \hline 0 & - & 0 & + & -\end{array}\right]\left[\begin{array}{c} v_1\\ v_2\\ v_3\\ \hline v_4\\ \hline v_5\end{array}\right]=0. \qquad (1.41)$$

According to (1.13) we assume $v_1 = +$. The set of a priori possibilities for the first three variables $(v_1, v_2, v_3)$ can thus be diagrammed as follows:



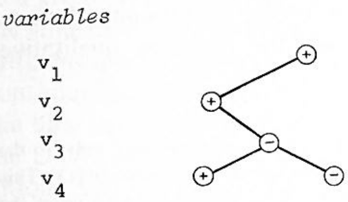

Only the branch



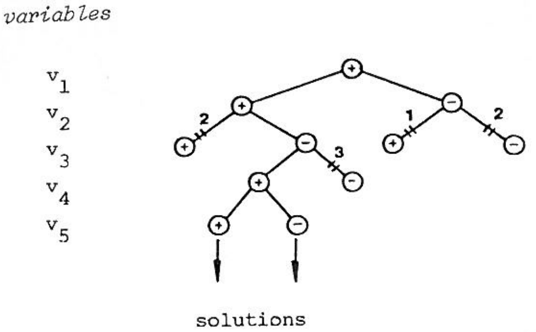

corresponds to a solution of the subsystem

$$\begin{bmatrix} - & + & - \\ 0 & + & + \end{bmatrix}\begin{bmatrix} v_1\\ v_2\\ v_3\end{bmatrix}=0,$$

thus the two following branches:




form the reference set for the second step. Pursuing along this way we can represent the construction of the qualitative solutions of (1.41) by the following branch and bound diagram:




where $0 \overset{k}{-\!\!\#\!\!-} 0$ denotes a branch eliminated by the $k$th relation. Thus (1.41) admits two solutions:

$$(+\quad +\quad -\quad +\quad +)' \quad \text{and} \quad (+\quad +\quad -\quad +\quad -)'.$$

These are obtained by testing 8 a priori possibilities whereas the basic eliminiation principle would require the testing of $2^4 = 16$ a priori combinations.

It must be noted here that the efficiency of the procedure is not independent of the variables ordering [16, pp. 76–78]. Maximal efficiency will be achieved if, when setting the system into the form (1.37), one seeks to minimize successively the number $p_i$ of columns of the matrix $H_{ii}$, $i = 1,2,\ldots, l$.

To give an idea of the gain in efficiency of the proposed recursive procedure as opposed to the basic Samuelson–Lancaster elimination principles, we can mention that for a 20 relations model the computation of the solutions for 5 exogenous variables was reduced from 30 minutes to 10 seconds on a UNIVAC 1108.

### *1.4. Illustration and additional information*

The purpose of this section is to provide an example of a comparative static study carried out by means of the techniques proposed in this first part.

A quite small macromodel is considered in order to allow the reader to check, by hand, the results presented. The relations, in their general formulation: $y = g(y, z)$, are given hereafter, together with the sign assumptions made about the derivatives of the behavorial relations:

(1) Private consumption: $\quad C = g_1\underset{(+)(+)}{(W, Y)}.$

(2) Private investment: $I = g_2(\underset{(+)}{P})$.

(3) Wages: $W = g_3(\underset{(+)}{Q})$.

(4) Corporate savings: $S = g_4(\underset{(+)}{P})$.

(5) Business taxes: $T = g_5(\underset{(+)}{Q})$.

Identities and definitional relations:

(6) Total production: $Q = C + I + \overline{G}$.

(7) Profits; $P = Q - W - T$.

(8) Non wage incomes: $Y = P - S$.

The only exogenous variable considered is $\overline{G}$, the public consumption.

The qualitative structure of the model, i.e. the sign content of the matrix $(\partial g/\partial z' \mid \partial g/\partial y' - I)$, is

$$\begin{array}{c} \\ 1\\3\\8\\6\\7\\4\\2\\5 \end{array}\begin{array}{c} \begin{array}{ccccccccc} \overline{G} & C & W & Y & Q & P & S & I & T \end{array} \\ \left[\begin{array}{ccccccccc} \cdot & - & + & + & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & - & \cdot & + & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & - & \cdot & + & - & \cdot & \cdot \\ + & + & \cdot & \cdot & - & \cdot & \cdot & + & \cdot \\ \cdot & \cdot & - & \cdot & + & - & \cdot & \cdot & - \\ \cdot & \cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & - & \cdot \\ \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & - \end{array}\right] \end{array} \quad (1.42)$$

Since the model is completely simultaneous there are no qualitative zeros.

From this basic qualitative information we set up the following classes of qualitatively linked variables:

$$\left.\begin{array}{l} \{Q, W, T\} \\ \{P, S, I\} \\ \{Y\} \\ \{C\} \\ \{\overline{G}\} \end{array}\right\} \text{ all links are positive.}$$

The resulting aggregated system characterized by the matrix

$$\begin{array}{c} \\ 1 \\ 6 \end{array}\begin{array}{c} \begin{array}{ccccc} \overline{G} & Q & Y & C & P \end{array} \\ \left[\begin{array}{ccccc} \cdot & (+) & (+) & (-) & \cdot \\ (+) & (-) & \cdot & (+) & (+) \end{array}\right] \end{array}$$

admits the 11 qualitative solutions

| | $\overline{G}$ | $Q$ | $Y$ | $C$ | $P$ |
|---|---|---|---|---|---|
| 1 | + | + | + | + | + |
| 2 | + | + | + | + | − |
| 3 | + | + | − | + | + |
| 4 | + | + | − | + | − |
| 5 | + | + | − | − | + |
| 6 | + | + | − | − | − |
| 7 | + | − | + | + | − |
| 8 | + | − | + | − | + |
| 9 | + | − | + | − | − |
| 10 | + | − | − | − | + |
| 11 | + | − | − | − | − |

These outcomes seem not very relevant. With respect to the generality of the assumptions made this is, however, not surprising. Thus, to obtain more significant results one has to consider further information.

The determinant of the matrix $A$ corresponding to the Jacobian $\partial h/\partial y'$ $(= \partial g/\partial y' - I)$ of the model is qualitatively undetermined. [7] Assuming the model represents the equilibrium relations of some underlying dynamic process, stability conditions would imply a given sign for the determinant $|\partial h/\partial y'|$. Our first additional constraint is thus

$$C_1: \quad |A| = \text{sign}\left[(-1)^8\right] = +.$$

From Cramer's rule: $x_i |A| = |A_i|$, where $A_i$ is the matrix obtained by substituting $-b(= -\partial h/\partial\alpha)$ to the $i$th column of $A$, an impact multiplier $x_i$ $(= \partial y_i/\partial\alpha)$ will be qualitatively determined under $C_1$ when the sign of the determinant $|A_i|$ is unambiguously determined from the basic qualitative assumptions. [8] This is for instance the case of the impact on $Q$, for which it is easy to check that the corresponding determinant $|A_i|$ is positive. Thus, remembering the qualitative links, imposing $C_1$ leads us to set

$$\frac{\partial Q}{\partial G} > 0, \qquad \frac{\partial W}{\partial G} > 0 \quad \text{and} \quad \frac{\partial T}{\partial G} > 0.$$

As shown by Lancaster [13, p. 288], by means of linear combinations, quantitative a priori information permits sometimes to define a new qualitative relation which can be added to the system analyzed. This technique enables us to take into account the 2 hypothesis

$$C_2: \quad 0 < \partial g_4/\partial P < 1,$$

$$C_3: \quad 0 < \frac{\partial g_3}{\partial Q} + \frac{\partial g_5}{\partial Q} < 1.$$

By considering, in addition, the exact value $(-1)$ of the diagonal elements of the Jacobian, as well as that of the coefficients of the definitional relations (7) and (8), we obtain the two new qualitative

[7] The determinant $|A|$ is undetermined if and only if the system $Ax = 0$ admits at least one non-zero solution [13]. This can be checked by means of an algorithm based upon the iterative qualitative aggregation principle [16, Chapter 4, Section 4.4].

[8] As for the qualitative determination of $|A|$, this can be checked by means of an algorithm based upon the iterative qualitative aggregation principle [16, Chapter 4, Section 4.4].

relations:

$$(9): \quad (+)v_P + (-)v_Y = 0,$$

$$(10): \quad (+)v_Q + (-)v_P = 0.$$

The relation (9) is obtained by substracting, in the Jacobian (1.42), row 4 from row 8, whereas the relation (10) is obtained by substracting rows 3 and 5 from row 7.

In order to study the implications of the constraints $C_2$ and $C_3$, one has simply to complete the system $Hv = 0$ with the two relations (9) and (10). This completed system can then be analyzed by means of the techniques described hereabove. Along the same line the implications of the constraint $C_1$ could be studied by adding to the system the relation:

$$(11): \quad (+)v_G + (-)v_Q = 0,$$

which ensures a positive link between $v_G$ and $v_Q$.

In our simple example this is indeed useless, since the supplementary relations (9) and (10) obviously imply:

$v_P L_+ v_Y$ and $v_Q L_+ v_P$.

By transitivity of the relation $L$, it follows that all variations are positively linked. Thus, under the constraints $C_1$, $C_2$ and $C_3$, it follows from the qualitative structure of the model, that all multipliers $\partial y_i / \partial G$ are positive.

## 2. The geometric approach

### 2.1. Intervals of knowledge and polytopes

If you have good enough data about an economy, computer facilities and a graduate student in econometrics who will agree to work for you, you can ask him, for instance, "What is the marginal propensity to consume (or to invest) with respect to a given variable for a given period of time?". If the student is conscientious, he will try to estimate different specifications and functional forms of the consumption (or investment) function, will select the best one according to criteria such as the $R^2$, the D.-W. statistic and some other validation conditions, and will give his conclusion as an estimate, a numerical value matched with a standard error coefficient. By repeating this experience several times with various students in different universities, and with more or less observations about the phenomenon, you will certainly obtain a wide range of results and conclusions about the marginal propensity to consume. And if you have no a priori reason to prefer one particular conclusion, your information about the marginal propensity to consume will be summarized by an appropriate interval which, for convenience, is assumed to be closed:

$$a^0 \leqslant a \leqslant a^*.$$

Exact quantitative knowledge corresponds then to the case $a^0 = a^*$, while $a^0 \neq a^*$ corresponds to more qualitative or imprecise situations.

More generally the specification of an econometric model of $m$ relations between $m$ endogenous variables $y$ and $k$ exogenous or predetermined variables $z$:

$$h(y; z) = 0 \Leftrightarrow h_i(y; z) = 0, \quad i = 1, 2, \dots, m,$$

normally gives rise to several sources of imprecision. One is due to the choice of the variables and the nature of the relations retained to define endogenous variables (i.e. accounting or definitional identities and behavioural relations). Others come from the necessary choice of a given parametric function for each behavioural relation, from identification and estimation problems linked with such relations, as well as with the latent distortion inherent to aggregated economic variables not directly observed.

There are essentially two ways enabling you to take into account imprecision. The first is the traditional probabilistic approach using confidence ellipsoids or intervals, which are based on the specification of a random vector of errors normally distributed, as well as on asymptotic considerations permitting to overcome difficulties due to non linearities. Although conceptually rigorous, this approach is not always very satisfactory when applied to non experimental situations, such as economic phenomena. In particular, asymptotic considerations are often relevant to fictitious situations with no practical feasibility.

To describe the second way, let us first consider (2.1) in its linearized form:

$$A\,\Delta y + B\,\Delta z = c \tag{2.2}$$

where $A$ is the Jacobian matrix $\partial h / \partial y'$, $B = \partial h / \partial z'$, $c$ an $m$ vector of residual terms, $\Delta y = y - y_0$, and $\Delta z = z - z_0$, and $(y_0; z_0)$ being given, for instance, by the last disposable observation. Setting

$$x=\begin{bmatrix}\Delta y\\ \Delta z\\ v\end{bmatrix},\qquad D=\begin{bmatrix}A & B & -c\\ 0 & I & -\gamma\\ 0 & 0 & 1\end{bmatrix}$$

and

$$b=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

(2.2) can be written in the following canonical form:

$$Dx=b, \tag{2.3}$$

where $v=1$ is an auxiliary variable and $\Delta z=\gamma$. Without loss of generality the $n\times n$ matrix $D$ (with $n=m+k+1$) is assumed to have all its diagonal elements equal to one.

Coming from various sources such as econometric estimations or a priori informations, we suppose that our knowledge about the model takes the form of a set of intervals $d_{ij}^0 \leqslant d_{ij} \leqslant d_{ij}^*$ for all the coefficients of the $D$ matrix:

$$\mathcal{D}=\{D \mid D^0 \leqslant D \leqslant D^*\}. \tag{2.4}$$

$\mathcal{D}$ is obviously a parallelotope in the parameters space, compact (closed and bounded) and convex.

This way of specifying economic knowledge by intervals is a general and very soft one, particularly to express differences in imprecision of knowledge about the past, and knowledge about the future.

Now, the problem to be faced is the study of significant properties of the geometric figure generated in the $n$-dimensional Euclidian space ($\mathbb{R}^n$ for convenience) by the set

$$P=\{x\in\mathbb{R}^n \mid Dx=b,\ D\in\mathcal{D}\}. \tag{2.5}$$

$P$ will be called a *polytope* in $\mathbb{R}^n$ [20, Chapter IX], and several questions about $P$ come immediately to mind, e.g. its projections with respect to the $x$-axes or on a selected plane corresponding to a couple $(x_i,x_j)$ of variables, and the particular points of $\mathcal{D}$ which give rise to some particular points of $P$.

Before going on with the study of these properties, let us illustrate this approach for $n=2$:

$d_1$: $\quad x_1=a_1x_2+b_1$,

$d_2$: $\quad x_2=a_2x_1+b_2$

where $b_1$ and $b_2$ are assumed to take fixed values and $a_1^0 \leqslant a_1 \leqslant a_1^*$, $a_2^0 \leqslant a_2 \leqslant a_2^*$. The Figs. 2–5 give some particular cases which may occur.

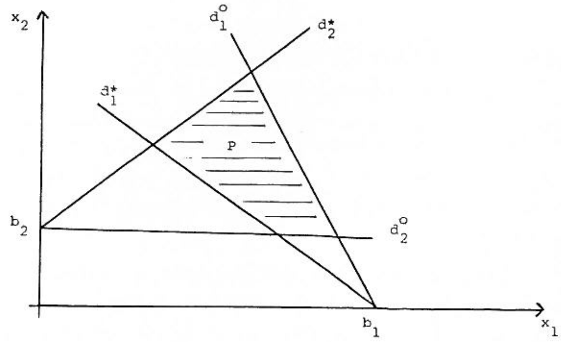


Fig. 2.

Figs. 2 and 3 illustrate the case of *regular polytopes, compact and connected*. In Fig. 2, $P$ is compact and convex. In Fig. 3, $P$ is non convex but it is still compact and connected.

Figs. 4 and 5 are relevant to *singular polytopes. P is no more compact and not necessarily connected.* Its projection with respect to the axes $x_1$, $x_2$, can be identical with the selected axes, and the only properties we can hope to bring out are qualitative properties such as: $x_1$ is always of the same (or opposite) sign as $x_2$.

Clearly the case of singular polytopes occurs when in the parameters space the closed set

$$K=\{D\in\mathcal{D} \mid |D|=0\} \tag{2.6}$$

is not empty, where $|D|$ is the determinant of matrix $D$. Fig. 6 illustrates this situation for $n=2$. $K=K_1\cup K_2$ is defined by the intersection of the set $\mathcal{Q}$ and the hyperbola

$$|D|=0 \Leftrightarrow a_1a_2=1.$$

## 2.2. Some properties of polytopes

Let us first consider the problem of eliminating the singular cases which may occur. For this pur-

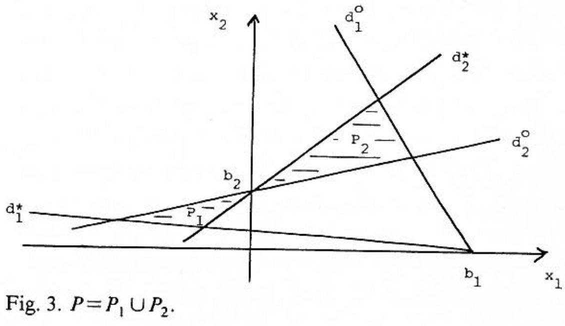


Fig. 3. $P=P_1\cup P_2$.

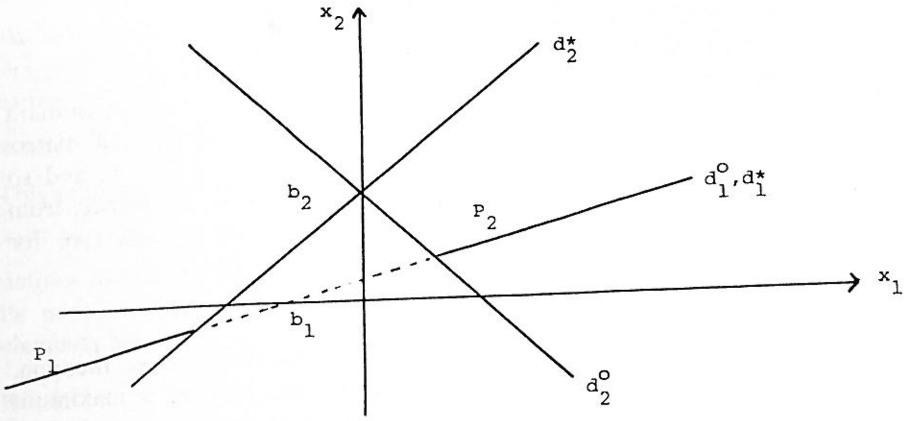



Fig. 4. $P = P_1 \cup P_2$.

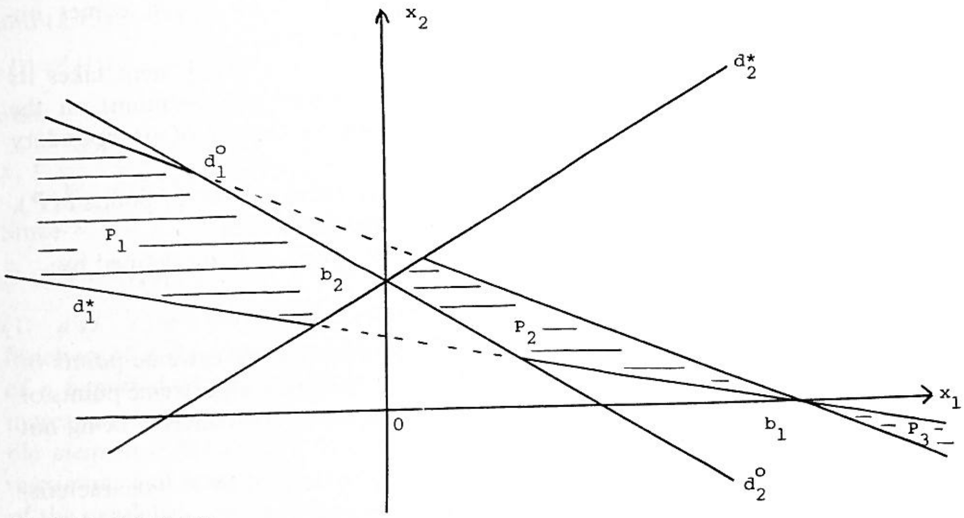



Fig. 5. $P = P_1 \cup P_2 \cup P_3$.

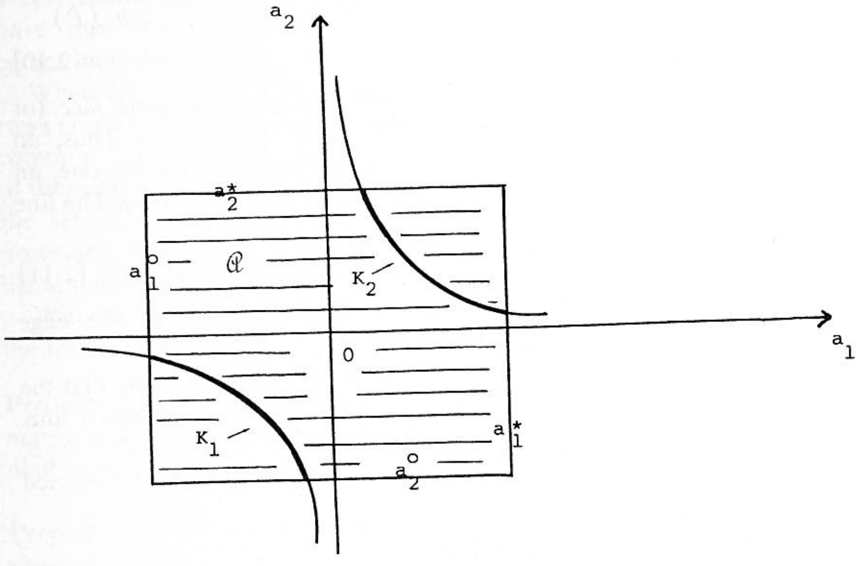



Fig. 6.

pose we will suppose from hereon that there are exactly $p < n^2$ elements $d_{ij}$ of matrix $D$ such that

$$d_{ij}^0 \leq d_{ij} \leq d_{ij}^*, \quad \text{with } d_{ij}^0 < d_{ij}^*,$$

i.e. $D$ contains $n^2 - p$ elements taking a fixed value. Then the parallelotope will admit exactly $2^p$ extreme points $D^{(l)}$, $l = 1, 2, \ldots, 2^p$, obtained by setting either $d_{ij} = d_{ij}^0$ or $d_{ij} = d_{ij}^*$ for each of the $p$ elements which can vary, called hereafter *variable elements*.

We will say that a point $D \in \mathbb{R}^{n^2}$ is *regular* iff $|D| \neq 0$ and *singular* iff $|D| = 0$.

Starting from the mean point

$$\bar{D} = \tfrac{1}{2}(D^0 + D^*) \tag{2.7}$$

assumed to be regular, we consider the $2^p$ matrices $B^{(l)}$ defined by

$$D^{(l)} = \bar{D} - B^{(l)}. \tag{2.8}$$

Hence, the existence of one singular point in the direction defined by $B^{(l)}$ supposes that equation

$$|\bar{D} - \epsilon B^{(l)}| = 0$$

has a real positive root $\epsilon$ such that $\epsilon \leq 1$. Since, for $\bar{D}$ regular, the equations $|\bar{D} - \epsilon B^{(l)}| = 0$ and $|I - \epsilon \bar{D}^{-1} B^{(l)}| = 0$ have the same roots, if $\rho^{(l)}$ is the greatest real eigenvalue of $\bar{D}^{-1} B^{(l)}$, it comes immediately that

$$K = \varnothing \text{ iff } \max_l \rho^{(l)} < 1 \tag{2.9}$$

where $K$, given by relation (2.6) above, is such that $K = \varnothing$ *iff $P$ is a regular polytope.*

When $P$ is singular, it is then possible to define reduction coefficients $r^{(l)} > 1/\rho^{(l)}$ for each matrix $B^{(l)}$ such that $\rho^{(l)} \geq 1$, which ensure the $\bar{D} - r^{(l)} B^{(l)}$ matrices to be regular points fulfilling condition (2.9).

It remains the problem of locating matrices $D^{(l)}$ for which such computations and modifications are necessary in order to have a regular polytope. For this purpose, let us recall that a determinant $|D|$ is a continuous function of its elements and that it cannot change sign without vanishing at a given point. Hence, the list of all the matrices we are looking for are given by the extreme points $D^{(l)}$ such that $|\bar{D}|$ and $|D^{(l)}|$ are of opposite signs. This list can easily be obtained from a simple test along the lines of the enumeration procedure for exploring parallelotope $\mathfrak{D}$ by modifying one variable element at a time which is described in Section 2.4 hereafter.

**Assumption.** *From hereon we shall assume that all the polytopes considered are regular.*

When $\mathfrak{D}$ is a compact set where the determinant $|D|$ does not vanish, all elements $d^{ij}$ of matrix $D^{-1}$ are continuous functions of $D \in \mathfrak{D}$, and so are the components $x_i$ of $x = D^{-1}b$. Hence, from topological properties of compact sets (see for instance [3, Chapters IV, V]), the set

$$P = \{x \mid x = D^{-1}b,\ D \in \mathfrak{D}\}$$

is *compact* and, from the Weierstrass theorem, each component $x_i$ reaches at least a maximum value and a minimum value on this set.

Now, taking into account the convexity of $\mathfrak{D}$ (and the continuity of $x$ on $\mathfrak{D}$), it comes immediately that $P$ is *connected*.

It comes also that the $x_i$ component takes its extreme values (maximum or minimum) on the *boundary* of $P$ given by the set of its boundary points $b(P)$.

In order to characterize this set of points $b(P)$, let us now give some definitions.

– The *characteristic points* of $P$ are defined by:

$$x^{(l)} = (D^{(l)})^{-1} b, \quad l = 1, 2, \ldots, 2^p,$$

where the $D^{(l)}$ matrices are the extreme points of $\mathfrak{D}$. As it will appear hereafter, all extreme points of $P$ are characteristic points, the converse being not true.

– An *edge* of $P$ will be defined by two characteristic points $(x^{(l)}, x^{(k)})$ such that

$$\lambda \in [0,1] \quad \text{and} \quad \lambda x^{(l)} + (1 - \lambda) x^{(k)} \in b_{lk}(P) \tag{2.10}$$

where $b_{lk}(P)$ is the boundary of some *face* (or subpolytope) of $P$ of dimension two. Thus, an edge can be seen as a face of dimension one, an extrem point as a face of dimension zero. The line given by

$$x(\lambda) = x^{(l)} + \lambda (x^{(k)} - x^{(l)}), \quad \lambda \in \mathbb{R} \tag{2.11}$$

will be called the *supporting line* of the edge $(x^{(l)}, x^{(k)})$.

For regular cases in $\mathbb{R}^n$ space, we know that the intersection of $(n-1)$ hyperplanes defines a line. Hence, by writing systems $D^{(l)}x = b$ and $D^{(k)}x = b$, having the same last $n-1$ rows, in the partitioned form

$$\begin{bmatrix} 1 & a' \\ c & H \end{bmatrix} \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{b} \end{bmatrix} \tag{2.12}$$

where $c$, $a$, $\bar{x}$, $\bar{b}$ are $(n-1)$-vectors, and $H$ an $(n-1)\times(n-1)$ matrix assumed to be regular, it comes:

$$\bar{x}=H^{-1}\bar{b}-H^{-1}cx_1. \tag{2.13}$$

This relation, along with

$$x_1=\lambda, \quad \lambda\in\mathbb{R}$$

defines the line passing through $x^{(l)}$ and $x^{(k)}$. Let us now suppose that $a$ is a vector of variable elements belonging to the parallelotope

$$\boldsymbol{a}=\{a \mid a^0\leqslant a\leqslant a^*\},$$

such that matrices $D^{(l)}$ and $D^{(k)}$ for instance are obtained at two extreme points of $\boldsymbol{a}$. From (2.12) and (2.13) it comes:

$$|D|=|H|(1-a'H^{-1}c)$$

and (2.14)

$$x_1=\frac{-a'H^{-1}b}{1-a'H^{-1}c}.$$

Since

$$D\in\mathfrak{D} \Leftrightarrow |H|(1-a'H^{-1}c)\neq 0,$$

$(1-a'H^{-1}c)\neq 0$ and $x_1$ is obviously a continous function of $a\in\boldsymbol{a}$, which can be seen as a branch of a hyperboloid. In particular, $x_1$ is monotonic, increasing or decreasing with respect to each variable element $a_i\in[a_i^0, a_i^*]$. Hence, $x_1$ will take its maximum and minimum values at extreme points of the paralellelotope $\boldsymbol{a}$. Moreover, to each edge of $\boldsymbol{a}$ corresponds an edge of $P$ and all these edges have the same supporting line. Generalisation comes immediately.

When $D^{(l)}$ and $D^{(k)}$ have only $n-r$ common rows $(r>1)$, the intersection of the corresponding common hyperplanes generates a hyperplane $\mathcal{H}_r$ of dimension $r$. Since $P$ is compact and connected, the set $\mathcal{H}_r\cap P$ is also compact and connected, necessarily bounded by a set of edges, each edge being bounded by two characteristic points of $P$.

Without other proof, we can then summarize the following properties for regular polytopes:

**Property 1** (P.1). *$(x^{(l)}, x^{(k)})$ is an edge of $P$ iff matrices $D^{(l)}$ and $D^{(k)}$ have exactly $(n-1)$ common rows*;

**Property 2** (P.2). *A component $x_i$ of $x$ takes its maximum and minimum values at extreme points of $\mathfrak{D}$ generating the characteristic points of $P$.*

**Property 3** (P.3). *In the $\mathbb{R}^n$ space, the boundary of $P$ is a polyhedron given by its characteristic points $x^{(l)}$, $l=1,2,\ldots,2^p$ and its edges $(x^{(l)}, x^{(k)})$.*

Furthermore, the dimension of $P$ in $\mathbb{R}^n$ space is obviously given by the number $k\leqslant n$ of relations containing variable elements.

### 2.3. Projections of polytopes

Let us first consider projections of $P$ with respect to the axes associated to $x$, axis $x_i$ for instance. We know, from the compactness and connectivity of $P$, that these projections are given by intervals of the form $[x_i^0, x_i^*]$ such that

$$x_i^0=\min_{D\in\mathfrak{D}} x_i \quad\text{and}\quad x_i^*=\max_{D\in\mathfrak{D}} x_i.$$

Optimal solutions to these problems, i.e. extreme points of $\mathfrak{D}$ and the corresponding characteristic points of the polytope $P$ can be obtained either by an enumeration procedure (see Section 2.4), or by an iterative procedure based on the sign of the partial derivatives:

$$\frac{\partial x_i}{\partial d_{hk}}=-d^{ih}x_k, \tag{2.15}$$

which are continuous functions of the variable elements of $D\in\mathfrak{D}$. Table 1 summarizes necessary conditions for $x_i$ to have minimum or maximum values on extreme points of $\mathfrak{D}$, which are also sufficient when $P$ is convex.

In particular, when the partial derivatives (2.15) are sign constants on $\mathfrak{D}$ (the case of and input–output open Leontief model, for instance) optimal solutions are immediately given. For more general cases, however, determination of the interval $[x_i^0, x_i^*]$ may require heuristic considerations using both the partial enumeration procedure and the necessary conditions above mentionned.

In order to study projecting of polytopes on planes, let $R$ be the projection of $P$ on the plane $\mathbb{R}^2$ corresponding, for instance, to the first two components $(x_1,x_2)$ of $x$. $R$ is obviously a compact and connected set, and its boundary gives rise

Table 1

| Sign of $\partial x_i/\partial d_{hk}$ | Search of $x_i^0$ | Search of $x_i^*$ |
|---|---|---|
| Positive | $d_{hk}=d_{hk}^0$ | $d_{hk}=d_{hk}^*$ |
| Negative | $d_{hk}=d_{hk}^*$ | $d_{hk}=d_{hk}^0$ |

to a polygon defined by the projection $P_R$ of a subset of the characteristic points $x^{(l)}$, $l=1,2,\ldots,2^p$ and by the projection of the edges linking these points. Since from P.1 above, we are able to test the existence of an edge linking two given characteristic points, the problem to be solved is the construction of the set $R$ without projecting all the characteristic points of $P$ but only an appropriate subset $C_R \supseteq P_R$ thereof.

Let $x^{(0)} = (D^{(0)})^{-1}b$ be a characteristic point of $P$. From $x^{(0)}$ we can define $p$ edges $(x^{(0)}, x^{(i)})$ such that $x^{(i)} = (D^{(i)-1}b$, matrices $D^{(i)}$, $i=1,2,\ldots,p$ being the $p$ extreme of $\mathfrak{D}$ adjacent to $D^{(0)}$, i.e. matrices obtained from $D^{(0)}$ by modifying only one variable element. Setting

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \bar{x}^{(i)} = \begin{bmatrix} x_1^{(i)} \\ x_2^{(i)} \end{bmatrix}, \quad i=0,1,\ldots,p,$$

projection of the points $x^{(i)}$ in the selected plane generates (for $p \geqslant 3$) a convex polygon $S$:

$$S = \left\{ \bar{x} \mid \bar{x} = \sum_{i=1}^{p} \lambda_i \bar{x}^{(i)}, \lambda_i \geqslant 0 \text{ and } \sum_{i=1}^{p} \lambda_i = 1 \right\}.$$

Hence, if $\mathring{S}$ and $\mathring{R}$ are the interiors of $S$ and $R$, respectively, it comes:

$$\bar{x}^{(0)} \in \mathring{S} \Rightarrow \bar{x}^{(0)} \in \mathring{R} \quad \left(\bar{x}^0 \notin P_R\right),$$

where $\bar{x}^{(0)} \in \mathring{S}$ iff

$$\exists \lambda_i > 0, \quad i=1,2,\ldots,p \quad \text{and} \quad \sum_{i=1}^{p} \lambda_i = 1$$

such that

$$\bar{x}^{(0)} = \sum_{i=1}^{p} \lambda_i x^{(i)}. \tag{2.16}$$

If $\bar{x}^{(0)} \notin \mathring{S}$, $\bar{x}^{(0)}$ will be called a *critical point* of $R$.

Clearly the set $C_R$ of all the critical points of $R$ is such that $P_R \subseteq C_R$, and its determination will probably enable us to eliminate a great number of characteristic points as interior points of $R$. In particular, if $P$ is a convex polytope, then $P_R = C_R$, the case $P_R \subset C_R$ occuring when projection $R$ masks a non convex part of $P$. For instance, we can have the following situations for $p=3$ (see Fig. 7).

In case (a) $\bar{x}^{(0)} \in \mathring{R}$ and it will not be retained as a critical point of $R$. In case (b) $\bar{x}^{(0)} \notin \mathring{S}$ and then it will be retained as an element of $C_R$.

Taking into account the fact that if two regular matrices $A = D^{(0)}$ and $B = D^{(i)}$ differ only by the value of one element $b_{hk} = a_{hk} + \Delta a_{hk}$, their inverses are such that

$$b^{ij} = a^{ij} - \frac{a^{ih}a^{kj}\,\Delta a_{hk}}{1 + a^{kh}\,\Delta a_{hk}},$$

where

$$1 + a^{kh}\,\Delta a_{hk} = \frac{|B|}{|A|},$$

it comes:

$$x_j^{(i)} = x_j^{(0)} - \frac{a^{jh}x_k^{(0)}\,\Delta a_{hk}}{1 + a^{kh}\,\Delta a_{hk}}, \quad j=1,2.$$

It is then possible to define a $2 \times p$ matrix

$$\Gamma = (\gamma_{ji}) = \left(x_j^{(i)} - x_j^{(0)}\right)_{i=1,2,\ldots,p,\, j=1,2}$$

such that conditions (2.16) above are equivalent to the existence of a strictly positive solution $\lambda > 0$ to the homogeneous linear system

$$\Gamma\lambda = 0.$$

From results on linear equations and inequalities (see for instance [14, p. 256]), we know that either $\Gamma\lambda = 0$ has a strictly positive solution $\lambda > 0$, or

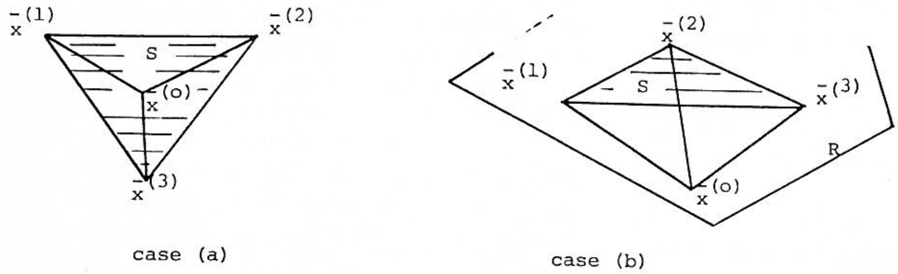



Fig. 7.

$y'\Gamma \geqslant 0$ has a solution. Hence, we must examine if there exist two numbers $y_1$, $y_2$ verifying the $p$ inequalities

$$y_1\gamma_{1i} + y_2\gamma_{2i} \geqslant 0, \quad i = 1,2,\ldots,p. \tag{2.17}$$

Taking the following partition of the index set $I = \{1,2,\ldots,p\}$:

$I_0 = \{i \mid \gamma_{1i} = 0\}$, $\quad I_1 = \{i \mid \gamma_{1i} > 0\ \}$ and

$I_2 = \{i \mid \gamma_{1i} < 0\}$,

(2.17) can be rewritten:

$$y_2\gamma_{2i} \geqslant 0, \quad \forall i \in I_0,$$

$$y_1 \geqslant -\frac{\gamma_{2i}}{\gamma_{1i}} y_2, \quad \forall i \in I_1,$$

$$y_1 \leqslant -\frac{\gamma_{2i}}{\gamma_{1i}} y_2, \quad \forall i \in I_2.$$

Three types of solutions can then be distinguished according to the value taken by $y_2$, which are summarized in Table 2.

If one of the conditions (a), (b) or (c) is verified, then $\Gamma\lambda = 0$ has no strictly positive solution, i.e. $\bar{x}^{(0)}$ is a critical point of $R$.

### 2.4. Algorithmic remarks

As it has been pointed out in Section 2.3 above, if two regular matrices $A$ and $B$ differ only by the value of one element $b_{hk} = a_{hk} + \Delta a_{hk}$, their inverses $B^{-1} = (b^{ij})$ and $A^{-1} = (a^{ij})$ are such that

$$b^{ij} = a^{ij} - \frac{a^{ih}a^{kj}\Delta a_{hk}}{1 + a^{kh}\Delta a_{hk}} \tag{2.18}$$

with the following relation between determinants:

$$|B| = |A|\left(1 + a^{kh}\Delta a_{hk}\right). \tag{2.19}$$

For such matrices $A$, $B$ ($A \in \mathcal{D}$, $B \in \mathcal{D}$), the edge $(A, B)$ of the parallelotope $\mathcal{D}$ will contain no singular points if $|A|$ and $|B|$ have the same sign as $|\bar{D}|$, where $\bar{D} = \frac{1}{2}(D^0 + D^*)$ is the mean point taken here as a regular reference point of $\mathcal{D}$.

Table 2

| Type of solution | Existence conditions |
|---|---|
| (a) $y_2 = 0$ | $I_1 \cup I_2 \neq \varnothing$ and $I_1 = \varnothing$ or $I_2 = \varnothing$ |
| (b) $y_2 = 1$ | $\max_{i \in I_1}\{-\frac{\gamma_{2i}}{\gamma_{1i}}\} \leqslant \min_{i \in I_2}\{-\frac{\gamma_{2i}}{\gamma_{1i}}\}$ and $\gamma_{2i} \geqslant 0$, $\forall i \in I_0$ |
| (c) $y_2 = -1$ | $\max_{i \in I_1}\{\frac{\gamma_{2i}}{\gamma_{1i}}\} \leqslant \max_{i \in I_2}\{\frac{\gamma_{2i}}{\gamma_{1i}}\}$ and $\gamma_{2i} \leqslant 0$, $\forall i \in I_0$ |

Hence, starting from an extreme point $A \in \mathcal{D}$ such that $\operatorname{sign}(|A|) = \operatorname{sign}(|\bar{D}|)$, it comes from relation (2.19) that edge $(A, B)$ does not have any singular points iff $a^{kh}\Delta a_{hk} > -1$.

Moreover, relation (2.18) can be used to calculate recursively the inverse $D^{-1}$ for all extreme point $D \in \mathcal{D}$, and thus the computation of only one initial inverse matrix is required: evaluation of characteristic points $x = D^{-1}b$ follows immediately.

Hence, it remains to describe an efficient enumeration procedure to obtain recursively all the extreme points of $\mathcal{D}$ by modifying one variable element at a time.

To this purpose, let $d$ be a $p$-vector containing the variable elements, such that

$$d^0 \leqslant d \leqslant d^*.$$

Starting from $d^{(1)} = d^0$, each term $d^{(l)}$ of the sequence $d^{(2)}, d^{(3)},\ldots, d^{(2p)}$ will be characterized by two integers $i_l$ and $k_l$ locating, respectively, the element $d_{i_l}$ which is modified when we go from $d^{(l-1)}$ to $d^{(l)}$, and the extreme point of $\mathcal{D}$ represented by $d^{(l)}$. In particular, $k_l$ is such that its binary expression is a $p$ ordered sequence of digits or bits 0,1 with

$$\text{bit } i = \begin{cases} 1 & \text{if } d_i^{(l)} = d_i^*, \\ 0 & \text{if } d_i^{(l)} = d_i^0, \end{cases}$$

$$i = 1,2,\ldots,p.$$

The selected procedure used to list the $2^p$ binary numbers is illustrated hereafter for $p = 3$ in Table 3.

For $p = 4$ we take the sequence obtained for $p = 3$, we rewrite the same sequence in the reversed order, add a 0 in the last position of the first $2^3$ numbers and a 1 to the last $2^3$ numbers, and so on.

For $l = 1,2,\ldots,2^p - 1$ we can then verify that $i_l$ is given by the following expressions:

$$i_l = 1 + \frac{\log n_l}{\log 2}, \quad \text{with}$$

$$n_l = \text{AND}(l, \text{COMPL}(l-1))$$

where COMPL$(l-1)$

gives the bit by bit logical complement of the binary expression of $l-1$ and AND$(l_1, l_2)$ is the integer obtained from the bit by bit logical product of the binary expressions of $l_1$ and $l_2$. AND( ) and COMPL( ) are standard functions in ASCII Fortran of the UNIVAC 1108 computer.

Table 3

| P | 1 | 2 | 3 |
|---|---|---|---|
| | 0 | 0 0 | 0 0 0 |
| | 1 | 1 0 | 1 0 0 |
| | | 1 1 | 1 1 0 |
| | | 0 1 | 0 1 0 |
| | | | 0 1 1 |
| | | | 1 1 1 |
| | | | 1 0 1 |
| | | | 0 0 1 |

### 2.5. *Illustration and practical considerations*

As an illustration, let us consider the simple macroeconomic model described in Section 1.4 above, which is written here in linearized form.

(1) Private consumption: $\Delta C = c_1 \Delta W + c_2 \Delta Y$.
Intervals: $0.7 \leqslant c_1 \leqslant 0.8$, $0.5 \leqslant c_2 \leqslant 0.6$.
(2) Private investment: $\Delta I = i \Delta P$.
Interval: $0.6 \leqslant i \leqslant 0.7$.

Table 4

| Hypotheses | $c_1$ | $c_2$ | $i$ | $w$ | $s$ | $t$ | $\Delta G$ |
|---|---|---|---|---|---|---|---|
| 0 | 0.7 | 0.5 | 0.6 | 0.6 | 0.4 | 0.08 | −1 |
| 1 | 0.8 | 0.6 | 0.7 | 0.7 | 0.6 | 0.10 | 2 |

(3) Wages: $\Delta W = w \Delta Q$.
Interval: $0.6 \leqslant w \leqslant 0.7$.
(4) Corporate savings: $\Delta S = s \Delta P$.
Interval: $0.4 \leqslant s \leqslant 0.6$.
(5) Business taxes: $\Delta T = t \Delta Q$.
Interval: $0.08 \leqslant t \leqslant 0.10$.
(6) $\Delta Q = \Delta C + \Delta I + \Delta G$.
Interval: $-1 \leqslant \Delta G \leqslant 2$.
(7) $\Delta P = \Delta Q - \Delta W - \Delta T$.
(8) $\Delta Y = \Delta P - \Delta S$.

Matrix $D$ contains 7 variable elements giving rise to $2^7 = 128$ extreme points of $\mathfrak{D}$ (and characteristic points of the polytope), each point being

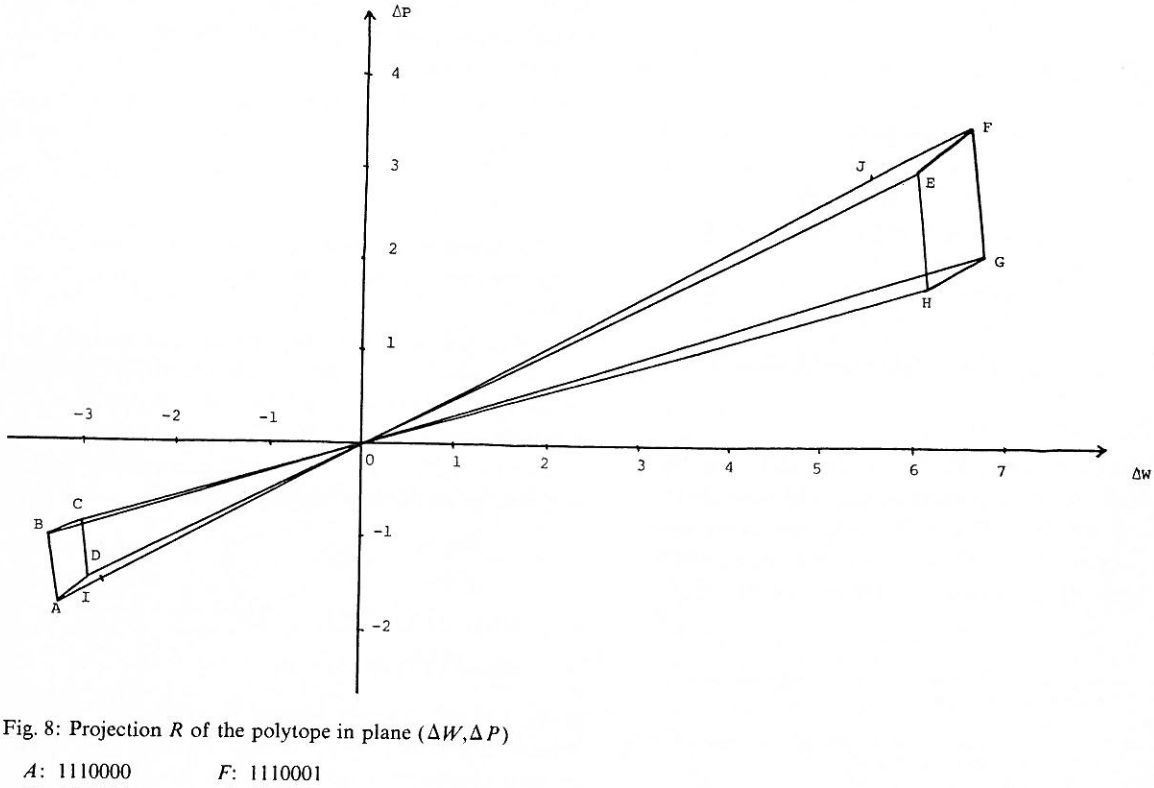



Fig. 8: Projection $R$ of the polytope in plane $(\Delta W, \Delta P)$

| | |
|---|---|
| *A*: 1110000 | *F*: 1110001 |
| *D*: 1111000 | *G*: 1111001 |
| *C*: 1111010 | *H*: 1111011 |
| *D*: 1110010 | *E*: 1110011 |
| *I*: 1110100 | *J*: 1110101 |

Table 5
Coordinates of the main points of projection $R$

| | Hypotheses | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $c_1$ | $c_2$ | $i$ | $w$ | $s$ | $t$ | $\Delta G$ | $\Delta Q$ | $\Delta C$ | $\Delta I$ | $\Delta W$ | $\Delta P$ | $\Delta S$ | $\Delta Y$ | $\Delta T$ |
| $A$: | 1 | 1 | 1 | 0 | 0 | 0 | 0 | $-5.53^{m}$ | $-3.29^{m}$ | $-1.24^{m}$ | $-3.32$ | $-1.77^{m}$ | $-0.71$ | $-1.06^{m}$ | $-0.44$ |
| $B$: | 1 | 1 | 1 | 1 | 0 | 0 | 0 | $-4.84$ | $-3.09$ | $-0.75$ | $-3.39^{m}$ | $-1.06$ | $-0.43$ | $-0.64$ | $-0.39$ |
| $C$: | 1 | 1 | 1 | 1 | 0 | 1 | 0 | $-4.39$ | $-2.77$ | $-0.61$ | $-3.07$ | $-0.88$ | $-0.35$ | $-0.53$ | $-0.44$ |
| $D$: | 1 | 1 | 1 | 0 | 0 | 1 | 0 | $-4.95$ | $-2.91$ | $-1.04$ | $-2.97$ | $-1.49$ | $-0.59$ | $-0.89$ | $-0.50^{m}$ |
| $E$: | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 9.90 | 5.82 | 2.08 | 5.94 | 2.97 | 1.19 | 1.78 | $0.99^{M}$ |
| $F$: | 1 | 1 | 1 | 0 | 0 | 0 | 1 | $11.06^{M}$ | $6.58^{M}$ | $2.48^{M}$ | 6.64 | $3.54^{M}$ | 1.42 | $2.12^{M}$ | 0.89 |
| $G$: | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 9.67 | 6.18 | 1.49 | $6.77^{M}$ | 2.13 | 0.85 | 1.28 | 0.77 |
| $H$: | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 8.77 | 5.54 | 1.23 | 6.14 | 1.75 | 0.70 | 1.05 | 0.88 |
| $I$: | 1 | 1 | 1 | 0 | 1 | 0 | 0 | $-4.56$ | $-2.54$ | $-1.02$ | $-2.74$ | $-1.46$ | $-0.88^{m}$ | $-0.58$ | $-0.37$ |
| $J$: | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 9.12 | 5.08 | 2.04 | 5.47 | 2.92 | $1.75^{M}$ | 1.17 | 0.73 |

Table 6

| Edges | | Compromise between objectives |
|---|---|---|
| $(G,F)$ | → | (max $W$, max $P$) |
| $(E,F)$ | → | (max $T$, max $Q$) |
| $(J,F)$ | → | (max $S$, max $Y$) |

represented by a binary number of 7 bits according to the convention described in Table 4.

For instance, the number 1101011 corresponds to $c_1 = 0.8$, $c_2 = 0.6$, $i = 0.6$, $w = 0.7$, $s = 0.4$, $t = 0.10$ and $\Delta G = 2$.

Selecting the projection plane ($\Delta W$, $\Delta P$), the resulting set $R$ is given in Fig. 8 , where we have reported only the significant points of the boundary and the interior points $D$ and $E$ associated with the projection of the polytope on the axis $\Delta T$.

As it can be seen in the Table 5, point $A$ generates minimum values for $\Delta Q$, $\Delta C$, $\Delta I$, $\Delta P$, and $\Delta Y$. Maximum values for these variables are obtained at point $F$ by moving along the edge $(A, F)$. For $\Delta W$, $\Delta S$ and $\Delta T$, these crucial edges are, respectively $(B,G)$, $(I,J)$, and $(D,E)$. Moreover, all these edges are generated by changing the value of $\Delta G$ from $-1$ to 2, and are going through the origin which is reached for $\Delta G = 0$.

From these results, $\Delta G$, can be seen as a variable defining the amplitude of the polytope; secondly, all the significant points in Fig. 8 are such that marginal propensities to consume $c_1$, $c_2$ and to invest $i$ are fixed to their maximum value. With respect to structural policy decisions, this corresponds to a consensus between firms and households, regardless of their possibly having other incompatible objectives.

This last remark suggests an interpretation of the significant edges of Fig. 8, as edges of compromise or negociation between conflicting objectives. Setting, for instance, $\Delta G = 2$, some crucial edges of compromise are listed in Table 6.

## 3. Conclusion

It is well admited, in economics, that knowledge derived from an entirely quantified model can be only approximative. Nevertheless, as opposed to the emphasis given on statistical and numerical methods devoted to precise quantitative information, less attention has been given, in economic modeling, to the treatment of more robust information like that defined by intervals.

We see two major reasons for this seemingly paradoxical tendency in economic modeling. The first is the *lack of efficient tools* for dealing with interval knowledge. The second is that, due to their generality, information by intervals *does not ensure precise and workable results.* For instance, it would not be significant to know that the variation $\Delta C$ of private consumption expenditure resulting from an increase $\Delta G$ of the public expenditures, will be comprised between $-\Delta G$ and $+2\Delta G$.

Let us note that qualitative and geometric models should not be considered as alternatives to quantitative models. They provide indeed knowledge of different nature and are therefore complementary. Thus for instance, the geometric approach focusses on extreme points whereas the common statistical procedures are mainly concerned with mean values or such representative figures.

Now, with respect to the two drawbacks pointed out, let us briefly expose our view of the scope and the future of qualitative and geometric methods.

Concerning the qualitative approach, we expect that the iterative qualitative aggregation procedure, which has been tested with systems of up to 200 relations, should allow to analyse a 1000 relation system within 15 minutes CPU time. This can be considered as a satisfactory performance from the point of view of efficiency. The problem lies, however, in the fact that significant results are generally meager as long as only the basic qualitative information is taken into account. As shown on our simple example in Section 1.4, additional information is usually necessary, in order to improve the undetermined results. Such information leads to the imposition of constraints on the qualitative system analyzed. When they take the form of qualitative relations these constraints can be treated by means of the same efficient qualitative techniques. The problem which remains is that of determining in each case, constraints which, while being not too restrictive, could improve significantly the qualitative results. Thus the development of the qualitative approach should focus on the possibility of seeking automatically, or at least quasi-automatically, efficient supplementary constraints. The overcoming of this difficulty would then allow a more systematical use of qualitative methods.

In order to give an idea of the actual scope of the qualitative approach, we can mention that a qualitative analysis of a 20 relation econometric model has been carried out without major difficulties in [17]. The same techniques have also permitted a more complex qualitative study of a model of about 170 relations in [16].

For the geometric approach the problem is quite different. Indeed the definition of the intervals is here flexible, so that an appropriate choice should always make it possible to obtain workable conclusions. The limitation of the approach is thus mainly that of the efficiency of the tools. It is obvious that the geometric illustration given in Section 2.5 is too simple to have other purposes than pedagogic ones. Up to now, we have some experiences with more sophisticated models of about 20 relations and 15 variable elements. But the practical feasibility of this approach is certainly compatible with large scale models for projections with respect to the axes. If we also require projections on selected planes, we are actually restricted by the number of variable elements generating the $2^p$ characteristic points of the polytope, since the projection procedure is based on an enumeration procedure, which needs about one minute CPU on UNIVAC 1108 computer for 2000 characteristic points. Other procedures are under study in order to construct directly projections of polytopes on selected planes without exploring all the characteristic points, and we hope to deal with models of more than 100 relations and 50 variable elements in less than five minutes CPU, including the automatic design of a projection by the computer.

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