The Fundamentals of Monotone Processes Reviewed Through an Inefficiency Measure

Gilbert Ritschard

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REVIEWED THROUGH AN INEFFICIENCY MEASURE*

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I. INTRODUCTION

The goal of any allocation mechanism is to improve the initial situation given by the endowments of the agents. For classical convex economies with price systems, the competitive process looks to be the *best* such mechanism. Indeed, from the New-Welfare Theorems of Arrow, Debreu, and Koopmans, we know that it achieves Pareto-efficient allocations. It is also well-known that the competitive mechanism is individually rational, i.e., the competitive equilibrium makes the position of each individual at least as good as his initial endowment (see, for instance, the nice book of Hildenbrand and Kirman [1976], where it is shown that the competitive equilibria belong to the core of the economy). Furthermore, the competitive mechanism is uniquely informationally efficient [Mount and Reiter, 1979; Jordan, 1982] which means, in particular, that it requires a minimal message space.

Despite all these nice properties of the competitive mechanism, other processes are of interest, especially for environments where the competitive allocation process does not work, or at least loses some of the above-mentioned properties. If we except the problems of nonconvexities due to indivisibilities, externalities, etc., these situations correspond mainly to the cases where there are no free prices (see, for instance, Chapter 7 in Balasko [1988]) or no prices at all. Think of planned economies, or of economies with public goods.

As far as decentralized processes are concerned, the agents must indeed get some information from the market, but the absence of a free price system here prevents the use of price signals. The messages sent to the agents must be of some other type and will generally take the form of quantity signals. Thus, the allocation mechanism without prices is commonly viewed as a system in which some central board proposes successively feasible allocations to the agents until an equilibrium is reached. At each step the propositions are naturally revised according to the agents' reac-

tion. In the well-known MDP [Malinvaud, 1970; Drèze and de la Vallée Poussin, 1971] process, for instance, the agents transmit to the central board their marginal rates of substitution evaluated at the proposed allocation.

Now, if we accept the idea that each agent is free to reject a proposed allocation, we can reasonably consider that he will do so for any proposition that makes him worse off compared with the previous one. The allocation mechanism can then only generate sequences of allocations along which the satisfaction of every agent is increasing or, at least, not decreasing. In that sense, such allocation mechanisms are called *monotone processes*.

Clearly, these processes based upon allocation signals are informationally much less efficient than mechanisms with price signals. Indeed, with m agents, the allocation space, i.e., the board's message space, has dimension $(m - 1)l$ as opposed to $l - 1$ for the price space. And this is without considering the agents' response space whose dimension, for instance, in the MDP process, equals $m(l - 1)$, i.e., the dimension of the total message space of the competitive mechanism. What about other properties?

The monotonicity requirement, which is expressed as an exchange axiom in Smale [1976], obviously implies the individual rationality of the mechanism. It is indeed a stronger property since it concerns every couple of successive allocation signals, while individual rationality expresses just the dominance of the final allocation over the initial endowments. It must be emphasized that the monotonicity property only makes sense for mechanisms that generate sequences of feasible allocations. This, for instance, is not the case of the competitive process.

From the point of view of social welfare, the crucial question is that of the accessibility of Pareto optima. On this point, monotonicity, together with the usual convexity and smoothness assumptions on preferences, ensures important properties. Smale [1973], for instance, has first demonstrated the accessibility and the stability of Pareto optima (see also Cornet [1981]), while Schecter [1977], in a very arduous paper, has established the finite length of monotone exchange curves.

The main goal of this note is to put forth the fundamentals of monotone processes. We therefore give a simplified and unified presentation of these results. For instance, we shall not consider Smale's problems of nonconvexities, nor shall we, contrary to Schecter, bother with the difficulties arising at the frontier of the commodity space. Our simplified setting consists then of a pure

exchange economy with the standard assumptions on preferences, and concerns only the interior behavior of continuous monotone processes.

The unified, and as a consequence also simplified, aspect of the presentation results from an extensive use of an allocation inefficiency measure which has first been introduced by Balasko [1982]. It is shown that the gradient of this measure defines a differentiable monotone process which establishes existence of such mechanisms. This measure is then shown to be a Liapounov function for monotone processes from which convergence to and stability of Pareto optima follow directly. Then, and this is the main point of the note, the inefficiency measure suggests a suitable substitution of variable that facilitates the demonstration of the finite length of the exchange curves generated by monotone processes.

Allocation mechanisms without prices are of special interest in planning problems involving production and public goods. In a concluding section, we then briefly explain how the approach followed in the paper easily extends to these cases.

II. THE FORMAL SETTINGS

We consider pure exchange economies with l commodities and m agents, but without price system. Let $x_i = (x_i^1, x_i^2, \ldots, x_i^l)$ denote a commodity bundle of agent i. The preferences of every agent i are then represented by a utility function $u_i : R^l \to R; x_i \to u_i(x_i)$, which is (1) differentiable up to any order; (2) differentiably increasing (i.e., $\partial u_i(x_i)/\partial x_i^j > 0$ for every $j = 1, 2, \ldots, l$); and (3) differentiably strictly quasi-concave, such that the set $u_i^{-1}([c, \infty])$ is strictly convex for every real number c.

Let $r \in R^l$ be the vector of total resources that are assumed constant. Then, $X = \{x = (x_1, x_2, \ldots, x_m) \in R^{lm} \mid \Sigma \, x_i = r\}$ denotes the set of the feasible allocations. The set X is obviously a smooth manifold of dimension $l(m - 1)$. The Pareto-efficient allocations in X, i.e., the allocations $x \in X$ for which there is no $x' \in X$ such that $u_i(x_i) \leq u_i(x_i')$ with at least one strict inequality for every i, form, in X, a submanifold of dimension $m - 1$. We denote this submanifold by P.

An *allocation process without prices* is a smooth vector field $\psi : X \to R^{lm}; x \to \psi(x) = (\psi_1(x), \psi_2(x), \ldots, \psi_m(x))$ with $\Sigma \, \psi_i(x) = 0$. The vector $\psi(x)$ indeed gives the direction of the change in the proposed allocation that occurs at the point x. The condition $\Sigma \, \psi_i(x) = 0$ ensures at each step that the resulting proposal remains feasible and unwasteful with respect to total resources.

Let the map $x:[0,\infty) \to X$; $t \to x(t)$ be the solution of the system of differential equations: $x'(t) = \psi(x)$, when the initial condition is $x(0) = \omega$, with $\omega \in X$ standing for the vector of initial endowments. This curve can be thought of as resulting from a sequence of small trades. Thus, we call it an *exchange or trade curve*.

A *monotone process* is a process ψ such that the satisfaction of every agent increases (one at least strictly) along every exchange curve. To put this formally, first note that, from the assumptions made on the utility functions u_i, the satisfaction of consumer i increases strictly in the direction $\psi(x)$ if and only if we have grad $u_i(x_i) \cdot \psi_i(x) > 0$. For grad $u_i(x_i) \cdot \psi_i(x) = 0$, strict quasi concavity implies a decrease in u_i, unless $\psi_i(x)$ is zero; i.e., unless agent i is not affected by the change in the allocation at x. Thus, a continuous allocation mechanism ψ is a monotone process if and only if at any point x of X the vector $\psi(x)$ belongs to the set,

$$C(x) = \{y = (y_1, y_2, \ldots, y_m) \in R^{lm} \mid$$
$$\Sigma\, y_i = 0,\ \text{grad } u_i(x_i) \cdot y_i > 0 \text{ or } y_i = 0, \text{ each } i\}.$$

It is readily shown [Smale, 1976] that $C(x)$ is a (neither open nor closed) convex cone and that we have $C(x) = \{0\}$ if and only if x is Pareto optimal. Assuming that some change occurs when x admits a Pareto superior allocation, i.e., $\psi(x) \neq 0$ if $C(x) \neq \{0\}$, this last property implies that the equilibria of monotone processes are given by the set P of the Pareto optima.

III. ALLOCATION INEFFICIENCY AND THE ASYMPTOTIC BEHAVIOR OF MONOTONE PROCESSES

The new approach followed in this note for studying the dynamics of monotone processes is based upon Balasko's [1982] inefficiency measure. In this section we first introduce the measure and its properties. We then give the direct consequences of these properties, namely, the existence of smooth monotone processes, convergence to Pareto optima, and the local asymptotic stability of these optima.

Given a point x of X, we consider the set of the allocations that dominate it; i.e., that are Pareto superior to x:

$$K(x) = \{\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_m) \in X \mid u_i(\bar{x}_i) \geq u_i(x_i), i = 1, 2, \ldots, m\}.$$

Clearly, for quasi-concave utility functions, $K(x)$ is a compact convex subset of X. We call it a *lens* because of its shape in the case of two goods and two agents.

Now, consider in X the Lebesgue measure of the lens $K(x)$, i.e., its volume. This measure, denoted $\mu(x)$, can be thought of as *the number* of allocations Pareto superior to x. In that sense, it reflects the degree of *inefficiency* of the allocation x.

Obviously, from the smoothness of the utility functions the inefficiency measure $\mu:X \rightarrow R_+$, is also smooth. Furthermore, we have

LEMMA 1. Assume a pure exchange economy with preferences represented by smooth, strictly increasing and strictly quasi-concave utility functions u_i. Then, the inefficiency measure $\mu(x)$ is such that

(a) $\mu(x) = 0$ if x is in P, and $\mu(x) > 0$ otherwise;

(b) for any exchange curve $t \rightarrow x(t)$ of a monotone process, $\mu'(t) < 0$ when $x(t)$ is not in P;

hence, since P is the set of equilibria of monotone processes, $\mu(x)$ is a Liapounov function for such processes;

(c) the vector fields $-$ grad $\mu(x)$ defines a smooth monotone process.

Proof of Lemma 1. For point (a), first note that, for strictly quasi-concave u_i's, a Pareto-efficient point x is the sole element of $K(x)$. This implies that $\mu(x) = 0$. The Lebesgue measure is by definition nonnegative. It then remains to show that $\mu(x)$ is nonzero outside the Pareto optima. For $\mu(x)$ to be nonzero, the lens $K(x)$ has to be of dimension $l(m - 1)$. This is the case if $K(x)$ has a nonempty interior. Here, note that the interior of the set $C(x)$ defined in Section II is the interior of the tangent cone of $K(x)$ at x. We then can, since $K(x)$ is compact and convex with our assumptions, equivalently check that the interior of $C(x)$ is not empty. This interior is given by the intersection of the open subspaces defined by the inequalities grad $u_i(x_i) \cdot y_i > 0$. Now, this intersection is not empty if the l hyperplanes grad $u_i(x_i) \cdot y_i = 0$ are not confounded. This is the case if at least two gradient vectors grad u_i are not collinear; i.e., if x is not in P. Part (a) is thus proved.

Point (b) is quite obvious. Indeed, for $x = x(t)$ and ψ being a monotone process, $x'(t) = \psi(x(t))$ lies in $C(x)$ by definition. Likewise, we have $x'(t) \neq 0$ when x is a nonoptimal allocation. Then, moving from x into the direction $x'(t)$ leads to a Pareto superior point $\bar{x} \neq x$ in $K(x)$. Then we have $K(\bar{x}) \subset K(x)$ which implies that $\mu(\bar{x}) < \mu(x)$. Thus, μ strictly decreases which proves part (b).

Finally, let us turn to part (c). Since from the smoothness of the u_i's μ is smooth, the function $g:x \rightarrow -$grad $\mu(x)$ is also smooth

on X. When $\mu(x)$ is not zero, i.e., from (a), outside P, it is always possible to move somewhere inside $K(x)$ and so reduce $\mu(x)$. Thus, $-\text{grad } \mu(x)$ has to be nonzero when x is not Pareto optimal. It then remains to show that $-\text{grad } \mu(x)$ belongs to the cone $C(x)$ for any x in X. Let $g(x) = (g_1(x_1), g_2(x_2), \ldots, g_m(x_m))$ denote the vector $-\text{grad } \mu(x)$. Clearly, we have $\Sigma g_i(x_i) = 0$. Furthermore, if $g_i(x_i)$ is not zero, then $-\text{grad } u_i(x_i) \cdot g_i(x_i)$ is strictly positive. Otherwise, indeed, a displacement in the direction $-\text{grad } \mu(x)$ would lead to a lower indifference surface for agent i, which is contradictory with the maximal decrease in $\mu(x)$. Thus, $g(x)$ lies in $C(x)$, and (c) is proved.

<div align="right">Q.E.D.</div>

It follows from properties (a) and (b) that the inefficiency measure μ is a Liapounov function for monotone processes. The characterization of the Liapounov function used here, however, differs somewhat from the usual one. Indeed, we have adapted the definition for the case of a continuum of equilibria that we have to deal with. One has then to be warned about a direct application of Liapounov's stability theorem as stated in Hirsch and Smale [1974, p. 193]. In our case the Liapounov function does not ensure the stability of an equilibrium x in P. It implies the stability of the connected subset P of X in the sense that, for each neighborhood V of P in X, there is a neighborhood U of P in X such that every trajectory starting in U remains in V.

Along the same way, we have to make precise the notion of asymptotic stability for the case of a continuum of equilibria. Indeed, the usual definitions, like the one given in Hirsch and Smale [1974, p. 186], also implicitly assume isolated equilibria. The natural extension [Smale, 1973] is

DEFINITION. Let P be a continuum of equilibria. Then $x \in P$ is *locally asymptotically stable* if P is stable (in the sense stated above) and if for every neighborhood V of x in P there is a neighborhood U of x in X such that every solution $x(t)$ with $x(0)$ in U is defined and converges to an equilibrium in V.

In other words, this means that any solution curve that starts near an equilibrium $x \in P$ converges to an equilibrium close to x.

We can now state the properties of monotone processes that are direct consequences of Lemma 1.

THEOREM 2. Assume a pure exchange economy with preferences represented by smooth, strictly increasing, and strictly quasi-concave utility functions u_i. Then,

(a) smooth monotone processes exist;

(b) let $x:[0,\infty) \to X; t \to x(t)$ be a monotone exchange curve starting at $x(0) = \omega \in X$; then, $\lim_{t \to \infty} x(t)$ exists and is a Pareto optimum;

(c) the Pareto-optimal allocations are locally asymptotically stable for monotone processes.

Proof of Theorem 2. Property (a) results from the existence of the inefficiency measure μ and from Lemma 1 (c).

From Lemma 1 (b) we have $\mu'(t) < 0$, for $x \notin P$. Thus, as $t \to \infty$, the function μ converges to its absolute minimum which is zero, i.e., $\lim_{t \to \infty} \mu(x(t)) = 0$. But this limit is equal to $\mu(\lim_{t \to \infty} x(t))$ which implies that $\lim_{t \to \infty} x(t)$ exists. Furthermore, it equals zero if and only if $\lim_{t \to \infty} x(t)$ is Pareto optimal. This ends the proof of part (b).

The stability of P follows from the existence of the Liapounov function μ. Next, consider a neighborhood V of an efficient allocation in P. Let U be any lens $K(x)$ such that $K(x) \cap P$ is a subset of V. Since any monotone exchange curve starting in $K(x)$ lies in $K(x)$, and converges to a point of P, it converges to an equilibrium in $P \cap U \subset V$. Part (c) is thus proved.

Q.E.D.

IV. FINITE LENGTH OF THE EXCHANGE CURVES

We have now established that monotone processes always lead to Pareto optima and that these optima are locally asymptotically stable. For the achievement of optima to be interesting from the economic point of view, it has to be realizable into a finite lapse of time. Mathematically, this property of finite time corresponds to the finite length of the trajectories that lead from any allocation x in X to Pareto optima. The following theorem is then essential to give economic significance to the results of the preceding section.

THEOREM 3. Assume a pure exchange economy with preferences represented by smooth, strictly increasing and strictly quasi-concave utility functions u_i. Then, a monotone exchange curve $[0,\infty) \to X, t \to x(t)$, with $x(0) = \omega$, has finite length for any $\omega \in X$.

Proof of Theorem 3.   The length of the exchange curve is given by

$$\text{length } x([0,\infty)) = \int_0^\infty \|x'(t)\|dt,$$

where $\|\cdot\|$ denotes the usual Euclidean norm. One has then to show that this integral is definite. We proceed (see Schecter [1977] for an alternative proof) through a parameterization of the length of the path $x([0,T])$ by the inefficiency measure of $x(T)$, i.e. the Lebesgue measure μ of the lens $K(x(T))$. Let $s : [0,\infty) \to R$, $\mu \to s(\mu)$, be this parameterization. This function can be thought of as the geometrical relationship between the path $x[0,T])$ and the volume reduction $\mu_0 - \mu$ of the lens, where $\mu_0 = \mu(x(0))$. Economically, it can be interpreted as giving the time needed to reach an allocation of inefficiency $\mu < \mu_0$.

Clearly, the function s is smooth on $(0,\infty)$, since $x(\cdot)$ and $\mu(\cdot)$ are themselves smooth. Furthermore, s is strictly decreasing.

The length of the exchange curve $x([0,\infty))$ is $s(0)$. Thus, the theorem will be proved if we establish that

$$s(0) = \lim_{M \to 0} \int_M^{\mu_0} |s'(\mu)|d\mu < +\infty.$$

This is done by showing that there exists a strictly positive ϵ such that

(a) $|s'(\mu)| \le k,$ if $\mu \ge \epsilon,$

(b) $|s'(\mu)| \le c\mu^{-\alpha}$, with $0 \le \alpha < 1,$ if $\mu < \epsilon,$

where c and k are finite positive constants.

Point (a) is true for every $\epsilon > 0$ since the map s is smooth on $(0,\infty)$.

For point (b), note first that $s'(\mu) = s'(t)/\mu'(t)$, where $s'(t) = \|\psi(x(t))\|$ and $\mu'(t)$ are the derivatives of s and μ with respect to the time t. Now, we have

(1) $|\mu'(t)| = -\text{grad } \mu(x(t)) \cdot \psi(x(t))$

$$= \cos \theta(t) \cdot \|\text{grad } \mu(x(t))\| \cdot s'(t),$$

where $\theta(t)$ is the angle between $-\text{grad } \mu(x(t))$ and $\psi(x(t))$. From Lemma 1, $|\mu'(t)|$ is strictly positive if $x(t)$ is not Pareto efficient which implies that $\cos \theta(t)$ is strictly positive. Let $\theta_{\max}$ denote the maximum of $\theta(t)$ on the portion of the exchange curve defined by

$\mu < \epsilon$ and $\|x(t) - \lim x(t)\| > 0$. From the strict quasi concavity of the utility functions u_i, $\cos \theta_{\max}$ is strictly positive. Since both grad $\mu(x)$ and $\psi(x)$ equal zero for x Pareto efficient, we have

$$|\mu'(t)| \geq \cos \theta_{\max} \cdot \|\text{grad } \mu(x(t))\| \cdot s'(t)$$

and then

(2) $\qquad |s'(\mu)| = \dfrac{s'(t)}{|\mu'(t)|} \leq (\cos \theta_{\max} \cdot \|\text{grad } \mu(x(t))\|)^{-1}.$

To establish point (b), we have then to study the relationship between $\|\text{grad } \mu\|$ and μ in the neighborhood of the optimum $\hat{x} = \lim_{t\to\infty} x(t)$. Since $\hat{x}$ is Pareto efficient (Theorem 2), we know, from Lemma 1, that $\mu(\hat{x}) = 0$ and grad $\mu(\hat{x}) = 0$. Thus, in a neighborhood of $\hat{x}$, a Taylor-Young expansion, respectively, to the second order for $\mu(x)$ and to the first order for grad $\mu(x)$, gives

$$\mu(x) = \tfrac{1}{2}\,{}^t(x - \hat{x})H(x - \hat{x})$$

$$\text{grad } \mu(x) = H(x - \hat{x}),$$

where H is the Hessian matrix of μ at x; i.e.,

$$H = \left(\frac{\partial^2 \mu}{\partial x_i \partial x_j}\,(\hat{x})\right) \qquad i,j = 1,2,\ldots, lm.$$

H is symmetrical. Furthermore, it is positive definite since μ reaches a minimum at $\hat{x}$. Thus, there exists an orthogonal matrix C such that $H = C\,\Lambda\,{}^tC$, where Λ is the diagonal matrix of the (nonnegative) eigenvalues λ_i, $i = 1,2,\ldots, lm$ of H. Let $z = (z_1, z_2, \ldots, z_{lm})$ denote the vector ${}^tC(x - \hat{x})$. Then

$$\mu(x) = \frac{1}{2}\,{}^tz\Lambda z = \frac{1}{2}\sum \lambda_i z_i^2$$

and

$$\|\text{grad } \mu\| = ({}^tz\Lambda^2 z)^{1/2} = \left(\sum \lambda_i^2 z_i^2\right)^{1/2}.$$

Since P is a submanifold of dimension $m - 1$ in $X \subset R^{lm}$ and grad $\mu(x)$ is nonzero for $x \notin P$, H has some strictly positive eigenvalues. Let $\lambda_{\min}$ be the smallest such nonzero eigenvalue. Then we have $\sum \lambda_i^2 z_i^2 \geq \lambda_{\min} \sum \lambda_i z_i^2 = \lambda_{\min} 2\mu$ and in a neighborhood of $\mu = 0$:

$$\|\text{grad } \mu\| \geq (2\lambda_{\min})^{1/2}\mu^{1/2}.$$

Together with the inequality (2) this gives

$$|s'(\mu)| \leqslant c\mu^{-1/2}, \quad \text{with } c = (\cos\theta_{\max} \cdot (2\lambda_{\min})^{1/2})^{-1}.$$

The angle $\theta_{\max}$ being strictly less that $\pi/2$ and $\lambda_{\min}$ being different from zero, the constant c is finite and positive. Point (b) is thus proved.

Finally, we then have

$$s(0) \leq c \int_0^\epsilon \mu^{-1/2}d\mu + k \int_\epsilon^{\mu 0} d\mu.$$

Clearly, the integrals in the right-hand side are definite, and so is $s(0)$. This completes the proof of Theorem 3.

<div align="right">Q.E.D.</div>

Remark. Theorem 3 ensures that wherever a monotone exchange curve starts, it converges in finite time to a Pareto optimum. It says nothing more, however, about the speed of convergence. From the equations (1) and (2) in the proof, we see that the length of the exchange path does not depend upon the length of the vectors $\psi(x)$. It is only determined by the angles $\theta(t)$ and by the length of the gradients of the inefficiency measure along the path. Thus, the shortest exchange path will be obtained when (i) the norms $\|\text{grad } \mu(x(t))\|$ are large and (ii) the vectors $\psi(x)$ are close to the gradients grad $\mu(x)$ so as to render the angles $\theta(t)$ small. This last condition then suggests that the process $-\text{grad } \mu$ should be among the more efficient ones.

V. Economies with Production and Public Goods

Until now, we have considered monotone processes in pure exchange economies only. Note, however, that the results can be extended to economies with production and public goods without great difficulties. We give just a few indications about it.

First, we characterize an economy with public goods and production. We consider h public goods. Let $x_p, x_i \in R^{l+h}$ denote a commodity bundle of consumer i. The amounts x_p of public goods in these bundles must be the same for each consumer i. The preferences are then represented by utility function $u_i: R^{l+h} \to R$, $(x_p, x_i) \to u_i(x_p, x_i)$. We still assume that the functions u_i are smooth, differentiably increasing in each argument and strictly quasi concave. Concerning production, we consider n firms. Let $y_j \in R^l$

denote the net output of private goods of firm j and $y_{jp} \in R^h$ its output of public goods. The efficient production set is defined by n implicit functions $f_j(y_{jp} - x_p, y_j) = 0, j = 1, 2, \ldots, n,$ where the functions $f_j : R^{l+h} \to R$, are smooth, nondecreasing in all the arguments and strictly convex.

Let

$$z = (x_p, y_p, x, y) \in R^{(1+n)h + (m+n)l}, \text{ with } y_p$$
$$= (y_{1p}, \ldots, y_{np}) \text{ and } y = (y_1, y_2, \ldots, y_n)$$

denote a *program* in that economy. For resources $r_p \in R_h$ in public goods and $r \in R^l$ in private goods the set of *feasible programs* is

$$Z = \{z \in R^{(1+n)h + (m+n)l} | x_p = \Sigma\, y_{jp} + r_p,$$
$$\Sigma\, x_i = \Sigma\, y_i + r, f_j(y_{jp} - x_p, y_j) = 0, \quad j = 1, 2, \ldots, n\}.$$

Clearly, Z is a smooth manifold of dimension $nh + (m + n - 1)l - n$. The study of monotone processes for economies with public goods and production as just described requires that the set Z be considered instead of the allocation set Z. Furthermore, the lenses to be considered are

$$K(z) = \{\bar{z} = (\bar{x}_p, \bar{y}_p, \bar{x}, \bar{y}) \in Z |$$
$$u_i(\bar{x}_p, \bar{x}_i) \geq u_i(x_p, x_i), \quad \text{each } i = 1, 2, \ldots, m\}.$$

Monotone processes are then defined by smooth vector fields $\psi : Z \to R^{(n+1)h + (n+m)l}; z \to \psi(z)$, such that $\psi(z)$ belongs to the set:

$$C(z) = \{w = (v_p, v_1, \ldots, v_n, w_1, \ldots, w_{m+n}) \in R^{(1+n)h + (m+n)l} |$$
$$v_i \in R^h, w_i \in R^l, v_p + \Sigma\, v_i = 0, \Sigma\, w_i = 0,$$
$$\text{grad } u_i(x_p, x_i) \cdot (v_p, w_i) > 0 \text{ or } (v_p, w_i) = 0,$$
$$\text{each } i = 1, 2, \ldots, m\}.$$

Now, to extend the approach followed in this paper, it remains to define the appropriate inefficiency measure. Consider the natural projection $\pi : R^{(1+n)h + (m+n)l} \to R^{h-ml}; z = (x_p, y_p, x, y) \to (x_p, x)$. Clearly, $\pi(K(z))$ is a compact subset of $\pi(Z)$. A legitimate inefficiency measure of a program z is then given by the Lebesgue measure μ_π of $\pi(K(z))$ as a subset of $\pi(Z)$. In the same way we define P_π as the natural projection of the set of Pareto optima with production. For establishing the properties of monotone processes, one then checks that the map $\mu_\pi : z \to \mu_\pi(z)$ has the same properties

with respect to P_π as the map μ has with regard to P. Similar demonstrations as those given in this note then follow.

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REFERENCES

Balasko, Yves, "Budget-Constrained Pareto Efficient Allocations: a Dynamic Story," *Journal of Economic Theory*, XVII (1982), 239–42.
——, *Foundations of the Theory of General Equilibrium* (Boston: Academic Press, 1988).
Cornet, Bernard, "Contributions à la Théorie Mathématique des Mécanismes Dynamiques d' Allocation des Ressources," Thèse de doctorat d'Etat ès sciences mathématiques, Université Paris IX-Dauphine, 1981.
Drèze, J., and D. de la Vallée Poussin, "A Tatonnement Process for Public Goods," *Review of Economic Studies*, XXXVII (1971), 133–50.
Hildenbrand, W., and A. Kirman, *Introduction to Equilibrium Analysis* (Amsterdam: North-Holland, 1976).
Hirsch, Morris W., and Stephen Smale, *Differential Equations, Dynamical Systems and Linear Algebra* (New York: Academic Press, 1974).
Jordan, J. S., "The Competitive Allocation Process is Informationally Efficient Uniquely," *Journal of Economic Theory*, XXVIII (1982), 1–18.
Malinvaud, Edmond, "Procedures for the Determination of a Program of Collective Consumption," *European Economic Review*, II (1970), 187–217.
Mount, Kenneth, and Stanley Reiter, "The Informational Size of Message Spaces," *Journal of Economic Theory*, VI (1979), 161–92.
Schecter, Stephen, "Accessibility of Optima in Pure Exchange Economies," *Journal of Mathematical Economics*, IV (1977), 197–216.
Smale, Stephen, "Global Analysis and Economics I: Pareto Optimum and a Generalization of Morse Theory," in *Dynamical Systems*, M. M. Peixoto, ed. (New York: Academic Press, 1973).
——, "Exchange Processes with Price Adjustment," *Journal of Mathematical Economics*, III (1976), 211–26.